The television program, The Weakest Link, recently came to the US from the UK. It has drawn a lot of interest because of the acerbic style of its presenter, Anne Robinson, but it also raises some interesting questions of strategy, that we address in this paper.

First, let us outline how the game is played. In round one, eight players are asked questions in turn, with each correct answer raising the pot. Incorrect answers send the pot back to zero. Before their question is asked, the next player can bank the money so far accumulated. At the end of each round, one player is voted off and eventually one player takes home whatever money has been banked over seven rounds, the last round counting double.

Let's consider how well the players did in e.g. the show aired on May 7, 2001. The winner took home $60,500. Since, however, 78 out of the 117 questions (doubling up the last round) were answered correctly, he could have taken home $78,000, if everyone had banked the $1,000 gained from each correct answer. If everyone had waited instead for a sequence of two correct answers (worth $2,500) before banking and then banked, the winner would have received $70,000. If the strategy had been to bank only after sequences of three/four/five correct answers in a row, then the winner would have gained $60,000/$50,000/$50,000 respectively. There were no sequences of six correct answers in a row.

The example above suggests that the best strategy might be to bank after every correct answer. In the paper below, we use mathematics (some combinatorics/probability theory) to substantiate this claim somewhat, although there are some strange twists in the tale.

The Problem.

We will model the above situation in the following way. Let us assume that the probability that a contestant answers a question correctly is $p$. For the time being we assume that this probability is the same for each contestant. Using the example above, $p$ might be taken to be $78/117 = 2/3$ - this seems typical for this game. In a round of $n$ questions (in the early rounds, $n$ is usually about 18), the amount banked by various strategies will be computed.

The author thanks Andrew Singer and Ralf Koetter for very helpful discussions and input on this topic. He was partially supported by NSF grant DMS 99-70184.
First, suppose each player banks after every correct answer. This is expected to occur $np$ times and will yield $\$1000np$.

Second, suppose each player banks after a sequence of two correct answers and only then. We compute the expected yield as follows. The outcome of a round can be given as a sequence of length $n$ of ones and zeros with one representing a correct and zero an incorrect answer. So, for instance, if $p = 2/3$ and $n = 3$, then we get 000 (all three answers incorrect) with probability $1/27$, 001 with probability $2/27$, ..., and 111 with probability $8/27$. Our strategy will lead to $\$0$ being banked with probability $11/27$ and to $\$2,500$ being banked with probability $16/27$ (corresponding to the cases of 011, 110, 111). Then our expected yield is about $\$1,480$. Compare this with the expected yield of $\$2,000$ if the first strategy had been pursued instead.

The question then is to compute, for general $p, n, r$, the expected yield if the players pursue a strategy of banking after every sequence of $r$ correct answers and only then. We will call this the $r$th strategy.

**The Computation.**

Let $p$ be the probability of a correct answer and $q = 1 - p$ the probability of an incorrect answer. Let $f(n, r, p)$ denote the expected number of occurrences of disjoint 11's of length $r$ in a binary sequence of length $n$. The amount banked after $r$ correct answers is currently given by $1v[r]$, where $v$ is the vector $[1000, 2500, 5000, 10000, 25000, 50000, 75000, 125000]$. Thus $v[r]f(n, r, p)$ will be the expected payoff from the $r$th strategy in a round of length $n$.

For instance, continuing our above example, $f(3, 2, p) = p^3 + 2p^2q = p^3(1 + 2q/p)$, since each of 111, 110, 011 has one occurrence of 11 and they occur with probability $p^3, p^2q, p^2q$ respectively. It is clear that $f(n, r, p)$ can be written in the form $p^nF_n(r, x)$, where $F_n$ is a polynomial of degree $n - r$ in $q/p$ and so in $x := 1 + q/p = 1/p$.

**Lemma.** Let $H_n(r, x) = nx^{n-1} - (n + 1)x^n - (n - r)x^{n+r-1} + (n - r + 1)x^{n+r}$.

Let $n \mod r$ be denoted $e$, so $e \in \{0, 1, 2, ..., r - 1\}$. Then

$$F_n(r, x) = (H_n(r, x) - H_e(r, x))/(1 - x^r)^2$$

**Sketch of Proof.** The identity $F_n(r, x) + F_{n+1}(r, x) + ... + F_{n+r-1}(r, x) = nx^{n-1}(\ast)$ was first experimentally observed and then proven. Fixing $p$ and $r$, we denote $f(n, r, p)$ by $E_n$. The identity $(\ast)$ is then equivalent to $E_{n+r-1} + pE_{n+r-2} + ... + p^{r-1}E_n = np^r$.

For simplicity, we explain why this holds in the case $r = 2$; the general case is similar. Consider a binary sequence of length $n + 1$. The probability of position $i$ ($1 \leq i \leq n$) being the start of a 11 is $p^2$ and so we expect $np^2$ such occurrences.

This counts 11's but we only want disjoint 11's. Every 111 or 1111 gives a discrepancy of 1 between $np^2$ and $E_{n+1}$. Every 11111 or 111111 gives a discrepancy of 2. And so on. This discrepancy is accounted for by $pE_n$, whence the formula.

If $1 \leq n < r$, then $E_n$ (and so $F_n(r, x)$) is clearly 0. The identity $(\ast)$ therefore determines $F_n(r, x)$ for all $n$. The formula for $F_n$ given was actually found by summing derivatives of geometric series. To prove its correctness once found simply takes mathematical induction using $(\ast)$.
We can now answer the main question. For given \( n \) and \( p \), for which \( r \) is \( v[r]p^nF_n(r, p^{-1}) \) (the expected yield from applying the \( r \)th strategy) greatest?

The following PARI gp program gives the answer (denoted \( j(n, p) \)) for any chosen \( n \) and \( p \).

\[
v=[1000,2500,5000,10000,25000,50000,75000,125000]
\]

\[
h(n,r,x)=n\times x^{(n-1)}-(n+1)\times x^{(n-r)}+(n-r+1)\times x^{(n+r)}
\]

\[
f(n,r,p)=p^n\times h(n,r,p^{\frac{n}{r}})/\{(1-p^r)^{\frac{n}{r}}\}^2
\]

\[
j(n,p)=a=0; b=0; for(r=1,8, if(v[r]*f(n,r,p)>a,a=v[r]*f(n,r,p); b=r, )); b
\]

The following commands are useful in understanding the ranges of values of \( p \) for which a given strategy is best. \( k(n, r, s) \) gives the crossover probability (almost always unique) for changing between the \( r \)th and \( s \)th strategies, and \( l(n) \) tabulates these values for a given \( n \).

\[
k(n,r,s)=\text{solve}(p=.001,.999,v[r]*f(n,r,p)-v[s]*f(n,s,p))
\]

\[
l(n)=\text{for}(r=1,7,\text{for}(s=r+1,8,\text{print}([r,s,k(n,r,s)])))
\]

Looking for instance at the case \( n = 18 \), we see that for \( p < 0.563 \), the 1st strategy gives the greatest expected yield. For \( 0.563 < p < 0.695 \), the 6th strategy is best. For \( p > 0.695 \), the 8th strategy is best.

This same pattern holds for all \( n \geq 16 \) and \( 8 \leq n \leq 13 \). For \( n = 14 \) and 15, there are new wrinkles to the tale. For instance, for \( n = 15 \), the 7th strategy actually beats the 8th strategy for \( p > 0.951 \). (Note, however, that this is artificial. It arises from the possibility of two runs of 7 ones but impossibility of two runs of 8 ones. Such runs are unheard of in practice and the model would break down since the limit of \$125,000\) winnings per round would be reached. This limit has so negligible an effect otherwise as not to change the optimal strategy.) The limiting solution, as \( n \) tends to infinity, ignoring the \$125,000 limit, is to go with the 1st strategy if \( p < 0.529 \), the 8th if \( p > 0.647 \), and the 6th otherwise.

**Simulations.**

The simulations come from the show itself. Below we list data from seven shows in May and June 2001.

<table>
<thead>
<tr>
<th>Date</th>
<th>( p )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<tbody>
<tr>
<td>5/7</td>
<td>0.667</td>
<td>60.5</td>
<td>78</td>
<td>70</td>
<td>60</td>
<td>50</td>
<td>50</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5/14</td>
<td>0.675</td>
<td>107.5</td>
<td>85</td>
<td>77.5</td>
<td>80</td>
<td>100</td>
<td>150</td>
<td>100</td>
<td>75</td>
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<tr>
<td>5/21</td>
<td>0.500</td>
<td>42</td>
<td>55</td>
<td>42.5</td>
<td>40</td>
<td>40</td>
<td>25</td>
<td>50</td>
<td>0</td>
</tr>
<tr>
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<td>0.574</td>
<td>71</td>
<td>70</td>
<td>65</td>
<td>30</td>
<td>40</td>
<td>50</td>
<td>50</td>
<td>75</td>
</tr>
<tr>
<td>5/28</td>
<td>0.667</td>
<td>81</td>
<td>82</td>
<td>82.5</td>
<td>80</td>
<td>90</td>
<td>100</td>
<td>100</td>
<td>75</td>
</tr>
<tr>
<td>6/4</td>
<td>0.653</td>
<td>74.5</td>
<td>81</td>
<td>70</td>
<td>75</td>
<td>90</td>
<td>125</td>
<td>50</td>
<td>75</td>
</tr>
<tr>
<td>6/11</td>
<td>0.587</td>
<td>83.5</td>
<td>74</td>
<td>70</td>
<td>80</td>
<td>70</td>
<td>125</td>
<td>250</td>
<td>75</td>
</tr>
</tbody>
</table>

Total 0.619 520 525 477.5 445 480 625 600 375 250
The above table gives the proportion of questions answered correctly, the amount (in thousands of dollars) actually won, and how much the winning contestant would have taken home if they had pursued the \( r \)th strategy (\( 1 \leq r \leq 8 \)).

Compare this with the theoretical expectations of the amount won (to the nearest thousand dollars) under each strategy based on the empirical values of \( p \):

<table>
<thead>
<tr>
<th>“p”</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.619</td>
<td>546</td>
<td>501</td>
<td>479</td>
<td>502</td>
<td>686</td>
<td>764</td>
<td>641</td>
<td>599</td>
</tr>
</tbody>
</table>

**Conclusions.**

Our assumptions would seem to support the 6th strategy, since \( p \) is typically in that range. Looking at the above data, the 5th and 6th strategies do seem to do best in the long run. The safest bet is the 1st strategy (a good choice for the cautious), which usually does better than the contestants’ chosen strategy and will pretty much guarantee at least $70,000 in winnings, but might not satisfy the ambitious. Attempting the 5th or 6th strategy is much more of a risk and will of course incur repeated derision by Anne Robinson on the rounds that nothing is banked. I thank the anonymous referee for running a simulation of 100,000 rounds with \( p = 0.619 \) (the average over all the seven shows) and observing that under the 6th strategy nothing would be banked for the whole show about 8% of the time. Another reason to bank regularly is because voting ties are resolved by the “strongest link”. If two players have the same scoring record, then the player who has banked more is regarded as stronger.

Voting strategies bring in many other issues. Recent shows suggest that being the best player can backfire on you near the end of the show, when players want to vote off their competition. For instance, on the last two shows (6/4 and 6/11) the strongest player was voted off midway with the second strongest player ultimately winning. It seems advisable to get some questions wrong (on purpose if necessary - just be sure to answer quickly). Also, make sure you vote for people certain to be voted off - revenge is commonly cited as a reason for choosing someone to vote for. Lastly, on at least three occasions, women have admitted to forming an alliance in advance (something the game rules allow) based on wanting a woman to be overall winner. This could even lead to a backlash whereby men vote off women for fear they will gang up against them.

Further work will consider mixed strategies (i.e. with a probability \( p_r \) of banking after \( r \) consecutive correct answers). Calculations so far suggest that the pure strategies (where each \( p_r \) is 0 or 1) yield the greatest expected winnings. The assumption of a constant \( p \) can also be relaxed (on 6/11 individual values of \( p \) varied between 0.438 and 0.750) and further classes of sequential strategies investigated. On the other hand, since many of the contestants have trouble with elementary mathematical problems (e.g. what is the square root of 121?), expecting them to implement some sophisticated scheme is maybe impractical. Thus banking after every correct answer (particularly for weaker contestants) seems a good idea.

The author recently learned that Paul Coe of Dominican University announced similar results in his January 7, 2002 talk at the Joint Mathematics Meetings in San Diego. See http://www.ams.org/amsmtgs/2049_abstracts/973-t1-634.pdf for his talk announcement, dated September 15, 2001. The current paper was received

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