Let $G$ be a group and $\Omega$ a set. (In this context, the members of $\Omega$ are often called “points”.) Suppose that we have a particular rule that determines a unique point in $\Omega$ whenever we are given a point in $\Omega$ and an element of $G$. Specifically, if $\alpha \in \Omega$ and $g \in G$ are our input data, this rule (denoted by a dot) determines a unique point $\alpha \cdot g \in \Omega$. (More formally, “dot” can be thought of as a function from the Cartesian product $\Omega \times G$ to the set $\Omega$.) In this situation, we say that $G$ acts on $\Omega$ if the following two conditions hold for all points $\alpha \in \Omega$ and all elements $g, h \in G$.

1. $\alpha \cdot 1 = \alpha$.
2. $(\alpha \cdot g) \cdot h = \alpha \cdot (gh)$.

The basic example of a group action is the case where $G$ is a subgroup of $\text{Sym}(\Omega)$, the symmetric group on $\Omega$, and dot is defined so that $\alpha \cdot g$ is just the result of applying $g$ to $\alpha$, so that $\alpha \cdot g = (\alpha)g$.

In general, if $G$ acts on $\Omega$, we can write $\theta_g$ to denote the function $\Omega \rightarrow \Omega$ induced by a group element $g$. In other words, the value of the function $\theta_g$ on the point $\alpha$ is precisely $\alpha \cdot g$. In symbols, we write $(\alpha)\theta_g = \alpha \cdot g$. The following is utterly routine:

**Lemma 1.** Let $G$ act on $\Omega$.

(a) $\theta_g \theta_h = \theta_{gh}$ for $g, h \in G$.

(b) The map $g \mapsto \theta_g$ is a group homomorphism from $G$ into $\text{Sym}(\Omega)$.

(c) The set $N = \{g \in G \mid \theta_g = \text{identity map} \}$ is a normal subgroup of $G$.

(d) If $N$ is as above, then $G/N$ is isomorphic to a subgroup of $\text{Sym}(\Omega)$.

We will not bother to write a detailed proof, but we make some remarks. The fact (asserted in (b)) that $\theta_g$ is a permutation of $\Omega$ is an immediate consequence of (a). Also, the normal subgroup described in (c) is called the **kernel** of the action. It is equal to the set of all of those elements $g \in G$ that fix all the points of $\Omega$.

An extremely important example of a group action is the action on the right cosets of a subgroup. If $H \subseteq G$, let $\Omega = \{Hx \mid x \in G\}$, the set of right cosets of $H$ in $G$. If $Hx$ is any right coset and $g \in G$, we can “translate” $Hx$ by $g$ by multiplying each element of $Hx$ on the right by $g$. This gives $(Hx)g = Hxg$, which is another right coset of $H$. We can thus define an action of $G$ on $\Omega$ by setting $(Hx) \cdot g = (Hx)g$, and it is a triviality to check that this is indeed an action.

Given a subgroup $H \subseteq G$, we define the **core** of $H$ in $G$ to be the subgroup obtained by intersecting all conjugates of $H$ in $G$. Symbolically, we write $\text{core}_G(H) = \bigcap H^x$, where $x$ runs over $G$. We have the following.

**Theorem 2.** Let $H \subseteq G$. Then $\text{core}_G(H)$ is the kernel of the natural action of $G$ on the right cosets of $H$. Also, $\text{core}_G(H)$ is a normal subgroup of $G$ contained in $H$ and it contains every other such normal subgroup.
Proof. First, suppose that \( N \triangleleft G \) and \( N \subseteq H \). Then \( N = N^x \subseteq H^x \) for all \( x \in G \), and thus \( N \subseteq \bigcap H^x = \text{core}_G(H) \).

If we take \( M \) to be the kernel of the action of \( G \), then of course \( M \triangleleft G \) and we claim that also \( M \subseteq H \). To see this, let \( m \in M \) and note that \( m \in Hm = (H \cdot m) = H \), where the last equality holds because \( m \) fixes the coset \( H1 = H \). We conclude from the first paragraph that \( M \subseteq \text{core}_G(H) \).

If \( g \in \text{core}_G(H) \) and \( x \in G \), then \( g \in H^x = x^{-1}Hx \), and thus \( xg \in Hx \). It follows that \( Hxg \subseteq HHx = Hx \), and thus \( (Hx) \cdot g = Hx \) for every element \( x \in G \). In other words, \( \text{core}_G(H) \) is contained in the kernel \( M \) of the action, and this completes the proof.  

The above result can be used to prove “nonsimplicity” theorems such as Corollary 4, below, which is often called the “\( n! \)-theorem”.

**COROLLARY 3.** Suppose \( H \subseteq G \) and \( |G : H| = n < \infty \). Let \( N = \text{core}_G(H) \). Then \( G/N \) is isomorphic to a subgroup of Sym(\( n \)), and thus \( |G : N| \) divides \( n! \).

**Proof.** We know that \( N \) is the kernel of the action of \( G \) on \( \Omega = \{Hx \mid x \in G\} \), which has \( n \) points. Also, by Lemma 1(d), we have that \( G/N \) is isomorphic to a subgroup of Sym(\( \Omega \)) \( \cong \text{Sym}(n) \).

**COROLLARY 4.** Suppose \( H \subseteq G \) with \( |G : H| = n \), where \( 1 < n < \infty \). If \( |G| \) does not divide \( n! \), then \( \text{core}_G(H) \) is a nonidentity proper normal subgroup of \( G \), and thus \( G \) is not simple.

**Proof.** Write \( N = \text{core}_G(H) \). Then \( N \triangleleft G \) and \( N \subseteq H < G \), so that \( N \) is proper. If \( N = 1 \), then \( |G| = |G : N| \) would divide \( n! \).

In addition to proving nonsimplicity theorems, group actions can also be used to count things. The key to this is the notion of an “orbit” of an action. Suppose \( G \) acts on \( \Omega \) and let \( \alpha \in \Omega \) be a point. Then the **orbit** of \( \alpha \) in this action, denoted \( O_\alpha \), is the set of all points of the form \( \alpha \cdot x \) as \( x \) runs over \( G \). The following is nearly trivial.

**LEMMA 5.** Let \( G \) act on \( \Omega \). If \( \alpha \in \Omega \), then \( O_\alpha \) contains \( \alpha \) and it is the only orbit of this action that contains \( \alpha \).

**Proof.** Certainly \( \alpha = \alpha \cdot 1 \in O_\alpha \). If also \( \alpha \in O_\beta \) with \( \beta \in \Omega \), we must show that \( O_\beta = O_\alpha \).

To see this, note that since \( \alpha \in O_\beta \), we can write \( \alpha = \beta \cdot g \) for some element \( g \in G \). Now if \( \gamma \in O_\alpha \), then for some element \( x \in G \) we have \( \gamma = \alpha \cdot x = (\beta \cdot g) \cdot x = \beta \cdot (gx) \), and thus \( \gamma \) lies in \( O_\beta \). This shows that if \( \alpha \in O_\beta \), then \( O_\alpha \subseteq O_\beta \).

To prove the reverse containment, we observe that \( \alpha \cdot g^{-1} = (\beta \cdot g) \cdot g^{-1} = \beta \cdot 1 = \beta \), and thus \( \beta \in O_\alpha \). By the result of the previous paragraph, we have \( O_\beta \subseteq O_\alpha \). Thus \( O_\beta = O_\alpha \), as required.

As usual, suppose that \( G \) acts on \( \Omega \). Since every point of \( \Omega \) lies in exactly one orbit, it follows that \( \Omega \) is the disjoint union of all the orbits. If the whole set \( \Omega \) consists of a single orbit, then we say that the action is **transitive**. (For example, the action of a group on the right cosets of any subgroup is transitive.) In general, if \( O \) is any orbit, then \( G \) acts transitively on \( O \).

If \( \alpha \in \Omega \), we write \( G_\alpha = \{x \in G \mid \alpha \cdot x = \alpha \} \). It is easy to see that \( G_\alpha \) is a subgroup of \( G \). It is called the **stabilizer** of \( \alpha \) in \( G \). For example, it is easy to check that in the action
of \( G \) on the right cosets of a subgroup \( H \), the stabilizer in \( G \) of the coset \( Hx \) is exactly the conjugate \( H^x \).

**THEOREM 6.** Let \( G \) act on \( \Omega \) and let \( \mathcal{O} \) be any orbit. Choose \( \alpha \in \mathcal{O} \) and let \( H = G_\alpha \) be the stabilizer of \( \alpha \) in \( G \). Then there is a bijection from the set of right cosets of \( H \) in \( G \) onto \( \mathcal{O} \).

**Proof.** First, note that \( \mathcal{O} = \mathcal{O}_\alpha \), so that the members of \( \mathcal{O} \) are exactly the points of the form \( \alpha x \) for \( x \in G \). Let \( \mathcal{C} = \{ Hx \mid x \in G \} \) be the set of right cosets of \( H \) in \( G \). We wish to define a function \( \varphi : \mathcal{C} \to \mathcal{O} \) by setting \( \varphi(Hx) = \alpha x \), but we must check that this is well defined. If \( Hx = Hy \), therefore, we must establish that \( \alpha x = \alpha y \). But \( y \in Hy = Hx \), and so we can write \( y = hx \) for some element \( h \in H \). Then \( \alpha y = \alpha h \cdot x = \alpha x \), where the last equality follows because \( h \in H = G_\alpha \), so that \( \alpha h = \alpha \). We can now define \( \varphi \) as above, and our task is to show that it is both surjective and injective.

For surjectivity, assume that \( \beta \in \mathcal{O} \). We must find \( Hx \in \mathcal{C} \) such that \( \varphi(Hx) = \beta \). We know, however, that \( \beta = \alpha x \), for some element \( x \in G \), and we have \( Hx \in \mathcal{C} \). Then \( \varphi(Hx) = \alpha x = \beta \), as required.

To prove that \( \varphi \) is injective, we suppose that \( \varphi(Hx) = \varphi(Hy) \) and we show that \( Hx = Hy \). We have \( \alpha x = \alpha y \), and thus \( \alpha = (\alpha x) x^{-1} = (\alpha y) x^{-1} = \alpha (yx^{-1}) \). In other words, \( yx^{-1} \in G_\alpha = H \). This yields \( H = Hyx^{-1} \), and from this, we deduce that \( Hx = Hy \), as required.

**COROLLARY 7.** (The Fundamental Counting Principle.) Let \( G \) act on \( \Omega \) and suppose that \( \mathcal{O} \) is a finite orbit. Then \(|\mathcal{O}| = |G : H|\), where \( H \) is the stabilizer of any point in \( \Omega \). In particular, if \( G \) is finite, then \(|\mathcal{O}| = |G|/|H| \) and this is a divisor of \(|G|\).

There are a number of important immediate consequences of the FCP.

**COROLLARY 8.** Let \( x \in G \), where \( G \) is finite. Then the size of the conjugacy class of \( x \) in \( G \) is equal to \(|G : C_G(x)|\).

**Proof.** The group \( G \) acts on itself by conjugation and the conjugacy class of \( x \) is exactly the orbit of \( x \) in this action. The stabilizer of \( x \) is \( C_G(x) \) and the result follows by the FCP.

**COROLLARY 9.** Let \( X \) be any subset of \( G \), where \( G \) is finite. Then the number of distinct conjugates of \( X \) in \( G \) is equal to \(|G : N_G(X)|\).

**Proof.** The group acts by conjugation on the set of all of its subsets. The orbit of \( X \) is the set of distinct conjugates of \( X \) and the stabilizer of \( X \) is \( N_G(X) \).

**COROLLARY 10.** Let \( H, K \subseteq G \) be finite subgroups. Then \(|HK| = |H||K|/|H \cap K|\).

**Proof.** Let \( K \) act on the set of right cosets of \( H \) in \( G \) by right multiplication. Then \( HK = \bigcup Hk \) as \( k \) runs over \( K \), and thus \( HK \) is the union of all right cosets of \( H \) in the orbit containing \( H \) under this action of \( K \). These cosets are disjoint and each has size equal to \(|H|\), and so we see that \(|HK| = n|H|\), where \( n \) is the size of the orbit. By the FCP, we know that \( n = |K : D| \), where \( D \) is the stabilizer in \( K \) of \( H \).
An element \( k \in K \) lies in \( D \) precisely when \( Hk = H \), and this happens precisely when \( k \in H \). This shows that \( D = H \cap K \), and thus \( n = |K : D| = |K|/|H \cap K| \). The result now follows.

The FCP also provides a tool for counting the number of orbits in an action. To explain this, we define the integer-valued function \( \chi \) on \( G \) corresponding to its action on a finite set \( \Omega \). This function, the permutation character associated with the action, is defined by the formula \( \chi(g) = |\{ \alpha \in \Omega \mid \alpha g = \alpha \}| \). In other words, the value of the character at \( g \in G \) is the number of fixed points of \( g \) in \( \Omega \). The following result, often incorrectly attributed to W. Burnside, says that the number of orbits of a finite group acting on a finite set is the average value of the associated permutation character.

**THEOREM 11.** Let \( G \) act on \( \Omega \), where both \( G \) and \( \Omega \) are finite, and suppose there are exactly \( n \) orbits in this action. Then

\[
    n = \frac{1}{|G|} \sum_{g \in G} \chi(g),
\]

where \( \chi \) is the associated permutation character.

**Proof.** Let \( \mathcal{P} \) be the set of ordered pairs \((g, \alpha)\) with \( g \in G \) and \( \alpha \in \Omega \) and such that \( \alpha g = \alpha \). Then for each element \( g \in G \), there are precisely \( \chi(g) \) members of \( \mathcal{P} \) having \( g \) as first entry and we see that \( |\mathcal{P}| = \sum \chi(g) \). Similarly, for \( \alpha \in \Omega \), exactly \( |G_\alpha| \) pairs in \( \mathcal{P} \) have second entry \( \alpha \), and this yields \( |\mathcal{P}| = \sum |G_\alpha| \), where this sum runs over \( \alpha \in \Omega \). We now have

\[
    \frac{1}{|G|} \sum_{g \in G} \chi(g) = \frac{|\mathcal{P}|}{|G|} = \sum_{\alpha \in \Omega} \frac{|G_\alpha|}{|G|} = \sum_{\alpha \in \Omega} \frac{1}{|O_\alpha|},
\]

where the last equality is a consequence of the FCP.

We compute the part of the final sum above contributed by the points lying in some particular orbit \( O \). For each point \( \alpha \) in \( O \), we see that \( O_\alpha = O \), and thus the contribution to the sum coming from \( \alpha \) is \( 1/|O| \). The total contribution from all points in \( O \) is thus \( |O| \) times \( 1/|O| \). In other words, each orbit contributes a total of 1 to the sum on the right, which is therefore equal to \( n \), as required.

We close with the following Corollary.

**COROLLARY 12** Let \( H < G \), where \( G \) is a finite group. Then there exists an element \( g \in G \) that lies in no conjugate of \( H \).

**Proof.** Let \( G \) act on the set \( \Omega \) of right cosets of \( H \) in \( G \). Since the stabilizer of the coset (point) \( Hx \) is \( Hx \), our problem is to find some element of \( G \) that fixes no point of \( \Omega \). We seek \( g \in G \), in other words, with \( \chi(g) = 0 \).

Our action is transitive, and so the average value of the permutation character \( \chi \) is 1. But \( \chi(1) = |\Omega| = |G : H| > 1 \) and \( \chi \) has an above-average value at the identity 1. It follows that \( \chi \) must have a below-average value at some element \( g \in G \). Thus \( \chi(g) < 1 \), but since \( \chi(g) \) is a non-negative integer, we deduce that \( \chi(g) = 0 \), as required.

We mention that the conclusion of Corollary 12 can fail if \( G \) is an infinite group.