Risk and Return Considerations in The Weakest Link

B. Ross Barmish and Nigel Boston

Electrical and Computer Engineering Department
University of Wisconsin
Madison, WI 53706

Abstract

The television game show, The Weakest Link, involves contestants making a sequence of decisions over time. Given the rules of the game and the process for accrual of payoffs over time, a number of authors have recognized that this show serves as a laboratory for assessment of human decision-making. To this end, by comparing theoretically derived gaming strategies with those actually used by the contestants, conclusions are drawn regarding the extent to which players' decisions are rational and consistent with the pursuit of optimality. The first main objective of this paper is to provide arguments that the models used in the literature to date may result in an erroneous impression of the extent to which contestants' decisions deviate from the optimum. More specifically, we first point out that previous authors, while concentrating on maximization of the expected value of the return, totally neglect the risk component; i.e., the expected return is considered while its variance is not. To this end, we expand the analysis of previous authors to include both risk and return and a number of other factors: mixing of strategies and so-called end effects due to fixed round length. It is seen that many strategies, discounted by previous authors as being sub-optimal in terms of maximization of expected return, may in fact be consistent with rational decision-making. That is, such strategies satisfy a certain "efficiency" requirement in the risk-return plane. To obtain this efficiency characterization of the game, the paper also includes results describing mean and variance of the return in closed form.

1. Introduction

The Weakest Link is a television game show that first appeared in Britain in August 2000. It rapidly became popular and was exported to more than two dozen foreign countries, with virtually the same set of rules as the original version. In the UK and US, the moderator, Anne Robinson, became notorious for her acerbic style, and was subsequently copied by local hosts elsewhere around the world. Except for repeats, the prime time version of the show no longer appears on U.S. television but there remains a shorter syndicated version of the show carried in the U.S. by a number of networks.

The game involves contestants answering general knowledge questions, with each correct answer raising the amount held in a temporary “pot.” However, before being asked a question, the contestant has to decide whether to “bank” what is already in the pot. If a decision to “bank” is made, the money in the pot is transferred to a permanent account and the pot is reduced to its starting value. Alternatively, the contestant may forgo the banking option. In this case, the contestant gambles the pot in the hope of increasing it by answering the next question correctly. If, however, an incorrect answer is given, the pot is lost and reset again to its starting value for the next contestant. Each round consists of a series of such scenarios, proceeding sequentially through the group. At the end of each round, one player is voted off by the others, until two players are left, at which point they play a round for double or triple the funds, and then answer a straightforward series of questions, the winner taking all, the other players receiving nothing.

1.1 More Detailed Description: To describe various versions of the game around the world we introduce the payoff vector

$$v = [v_1 \cdots v_m]$$

with v_k being the amount in the pot after a chain of k correct answers. The length of v is m and v_m is the limit on how much can be banked in one round. In the shows, typically the first round is 150 seconds long, decreasing by 10 seconds for each successive round. It takes about 7 – 8 seconds per question [6]. Below, we provide examples of the payoff vector for various versions of the game along with the empirically observed probability p of answering a question correctly. In each case, local currency is assumed.

Australia (see [3]): We have $p = 0.78$ and

$$v = [200, 500, 1000, 2000, 3000, 4500, 6000, 8000, 10000].$$

United Kingdom (see [6]): We have $p = 0.58$ and

$$v = [20, 50, 100, 200, 300, 450, 600, 800, 1000].$$

US Prime Time (see [2] and [7]): We have $p = 0.62$ and

$$v = [1000, 2500, 5000, 10000, 25000, 50000, 75000, 125000].$$

US Syndicated (see [7]): We have $p = 0.62$ and

$$v = [250, 500, 1000, 2500, 5000, 25000].$$

Strangely, reference [7] gives $v_6 = 12,500$, which disagrees with observation of the show.

France (see [5]): We have $p = 0.62$ and

$$v = [300, 750, 1500, 3000, 4500, 7500, 9000, 12000, 15000].$$

1.2 Strategies, Decisions and Social Issues: This game involves interesting questions of strategy, both in when to bank and whom to vote off, that have attracted interest from various disciplines, in particular mathematics, statistics, economics and management science. In addition, this game show provides an ideal laboratory to study human decision-making. The rules are well-defined and the stakes are high, something that is not easy to replicate elsewhere. The game also includes sociological and psychological aspects. In particular, in [1] and [8], the issue of discrimination in the voting is studied. Reference [5] also addresses the question of best voting strategy but just for the last but one round, using game theory and finding Nash equilibria. The players face a dilemma towards the end, in that they need strong players to help win the money but do not want to face someone stronger in the final round. The assumption, made in [8], that players will target stronger players towards the end, is shown to be false by data given in [5].

With the theory of bounded rationality providing context, more research has, however, been done on banking strategies; i.e., the objective is to provide a “strategy” which the contestant uses in making the decision whether to “bank” or risk losing accrued winnings for the opportunity to win a larger amount.

1.3 Literature: In the literature to date, the authors consider various strategies with objective being to maximize the expected value of the payoff or “return” of the game with infinite

round length implicitly assumed. This literature serves to motivate the more general risk-return analysis of this paper. Namely, we expand the earlier analysis from one-dimensional risk setting to the two-dimensional risk-return setting. In addition, we maintain the fixed round length framework of [2] and embellish the analysis to include consideration of strategy mixing and end effects.

With maximum return being the goal, both [3] and [6] consider strategies which are aimed at maximizing the expected payoff per question. In this regard, it is also interesting to note that the popular media have also picked up on results coming from this line of research. For example, following a press release by Dominican University, the unpublished work of Professor Paul Coe was cited in the New Scientist, USA Today and the London Times. Based on the assumption of infinite rounds and “pure” strategies, these papers restrict attention to the following fundamental question: If a player’s strategy is to “bank” after r correct answers, what is the optimal value of r ? The analysis of this so-called r -th *strategy* is also the focal point of this paper.

To this end, these authors consider the Australian and British versions, which are equivalent games since the payoff schemes are proportional. Using rather similar mathematics, they reach different conclusions. One reason for this difference may be that Australian competitors get about 78% of their questions correct whereas French, UK and US data from [2], [5], [6], and [8] indicate that players only get on average about 62% of questions correct. Accordingly, Australian competitors have incentive to try for longer chains of correct answers than elsewhere. The data of [6] finds that in the UK version, competitors become steadily worse at answering questions. Consequently, the authors in [6] recommend a change in strategy halfway through the game. In contrast, the data in [3] indicates that in the Australian game, competitors do consistently well throughout. [7], also with the British game, comes up with more sophisticated strategies depending on the success rate of the competitors.

In reference [2], the analysis of previous authors is expanded to account for the combinatorics associated with finite-length rounds. This author reaches similar conclusions to those in [3]

and [6], but with the caveat that trying for long chains of correct answers may be in general too risky. Consequently, he suggests that contestants play safe and bank every time they can. In [11], dynamic programming considerations are introduced into the analysis, allowing for consideration of more general strategies; i.e, not just pure strategies. The author also includes the fact that there is a limit on how much the players can bank in one round. Interestingly, it turns out that, apart from effects caused by this limit and end effects from needing to bank more frequently as the end of a round approaches, pure strategies are essentially the best ones.

1.4 Plan for This Paper: With the literature above providing motivation, this paper will also concentrate on finding the optimal value of r . However, one of our main contentions in this paper is that the approach in the existing literature, based on maximization of return (expected payoff), without accounting for the corresponding risk, can be quite misleading. That is, when we carry out the optimization in the more general risk-return framework, it is often the case that an “efficient strategy” is sub-optimal when risk alone is considered. In other words, based on previous literature in conjunction with TV game data, there is a temptation to erroneously classify certain strategies as inferior when in point of fact, they are perfectly rational from a risk-return point of view.

By taking both risk and return into account, our point of view becomes similar to that taken by much of the literature in areas such as economics and management science; e.g., see [4], [9] and [10]. That is, in the absence of a given utility function for a contestant, we can only narrow down a candidate set of strategies to those that are *efficient*; see Section 2 for details. Since it is generally not possible both simultaneously to maximize return and minimize risk, as seen in the sequel, a contestant might be well advised to try for the so-called *efficiency frontier* consisting of all risk-return pairs associated with non-dominated strategies. We study both return and risk and show that competitors might rationally behave quite differently from what the above papers propose.

Figure 1: **Efficiency Frontier**

2. Overview

As mentioned in the preceding section, this paper concentrates on the r -th strategy with characterization of risk-return tradeoffs being the objective. For a given strategy, if we take μ as the expected return and σ as the corresponding standard deviation, we define the so-called *efficiency frontier* as the set of achievable risk-return pairs (σ, μ) which are realizable and non-dominated. That is, a realizable risk-return pair (σ, μ) lies on the efficiency frontier for any other strategy with corresponding pair (σ', μ') , the following holds: If $\mu' > \mu$, then $\sigma < \sigma'$ and if $\sigma' < \sigma$, then $\mu > \mu'$.

In Figure 1, an illustration of the efficiency frontier is provided. In an economic context, it is arguable that a “rational” contestant will never select a strategy which results in an inefficient (μ, σ) point. Finally, it is also noted that the specific “operating point” which is selected cannot be pinpointed without further information about candidates’ utility functions.

With the paper's context described above, the plan for the sequel is as follows: In Section 3, we expand the the analysis of previous authors to account for *both* risk and return, round length, end effects and mixing of strategies; we obtain exact formulae for the expected return and its variance. In Section 4, we use these formulae to compute and graphically display efficiency frontiers for a number of cases of interest. This leads us to conclude that there is a much larger class of potentially optimal strategies than heretofore considered. Finally, in Section 5, some refinements to the theory are given. First, we consider the “practical” ramifications of a player picking one point on the efficiency frontier over another. Second, as the round length increases, we consider the extent to which returns are normally distributed.

3. Formulae for Risk and Return

Within the group of competitors, we assume that an approximately constant proportion p of the questions is correctly answered. For notational purposes we recall that v_i is the i -th component of the payoff vector v and that v_m is the limit on how much can be banked in one round. A round will consist of n questions. Similar to previous authors, we focus on the r -th strategy. However, in contrast to previous work, we also account for risk, end effects and mixing and examine strategies from an efficiency frontier point of view.

3.1 Formula for Return: In [2], an exact formula is given for the expected return for the r -th strategy applied to a round of length n and a constant probability p of a contestant successfully answering a question. We begin with this as a starting point and later refine our formulae to include end effects. Namely, with

$$H_n(r, x) = nx^{n-1} - (n+1)x^n - (n-r)x^{n+r-1} + (n-r+1)x^{n+r}$$

and $n \pmod{r}$ denoted e , so that $e \in \{0, 1, 2, \dots, r-1\}$, the expected return is $v_r p^n F_n(r, 1/p)$, where

$$F_n(r, x) = \frac{H_n(r, x) - H_e(r, x)}{(1 - x^r)^2}.$$

3.2 Formula for Risk: We likewise obtain a formula for the variance of the expected return for the r -th strategy applied to a round of length n , assuming a constant probability p of a contestant successfully answering a question. If $r = 1$, then this simply follows the binomial distribution and so the variance is $v_1^2 np(1-p)$. If $r = 2$, then the analysis becomes more complicated. The formula for variance in this case, which agrees with brute force computations for $n \leq 25$, is v_2^2 times

$$\frac{F_1(n, p) + (-1)^n F_2(n, p) + F_3(n, p)}{(1 - p^2)^4}$$

where

$$\begin{aligned} F_1(n, p) &= (n-1)p^2 - (n-2)p^3 - (6n-6)p^4 + (11n-24)p^5 + 31p^6 \\ &\quad - (11n+18)p^7 + (6n+4)p^8 + np^9 - np^{10}, \\ F_2(n, p) &= p^{n+2} + (2n-2)p^{n+3} - (8n+5)p^{n+4} + (10n+20)p^{n+5} \\ &\quad - 25p^{n+6} - (10n-14)p^{n+7} + (8n-3)p^{n+8} - 2np^{n+9} \\ F_3(n, p) &= -p^{2n+4} + 4p^{2n+5} - 6p^{2n+6} + 4p^{2n+7} - p^{2n+8} \end{aligned}$$

Note, however, that $F_2(n, p)$ and $F_3(n, p)$ will be small unless n is small and p close to one. We quantify this later. Neglecting F_2 and F_3 leads to the approximation $v_2^2 F_1(n, p)/(1-p^2)^4$ to the variance for $r = 2$. Note that this simplifies as v_2^2 times:

$$\frac{F_1(n, p)}{(1-p^2)^4} = \frac{n(p^2 + 2p^3 - 2p^4 - p^5)}{(1+p)^3} + \frac{(-p^2 - 2p^3 + 4p^4)}{(1+p)^4}.$$

For example, with $p = 0.5$, this says that the variance divided by v_2^2 is approximately given by $0.1019n - 0.0494$, which differs from the true variance by less than 0.0005 for $n \geq 10$. For each successive n , the difference approximately halves, so that for $n \geq 16$, the approximation error is less than 0.00001. For general p , the difference decreases by about p for each successive n . This approximation only becomes inaccurate when p approaches one. Even then, for $n \geq 9$, the largest error occurs at a value of p above 0.9 and for all $n \leq 24$ this error is less than 0.1.

A similar analysis for $r = 3, 4, \dots, 8$ leads to progressively more cumbersome expressions for the variance σ_r^2 for the r -th strategy. However, the corresponding approximations are more manageable. Namely,

$$\sigma_r^2 \approx v_r^2(n f_r(p) + g_r(p))$$

where

$$\begin{aligned}
f_r(p) &= p^r(1-p)(1-(2r+1)p^r + (2r+1)p^{r+1} - p^{2r+1})/(1-p^r)^3 \\
g_r(p) &= (-(r-1)p^r + rp^{r+1} + (3r^2 - 2r - 2)p^{2r} - 6r^2p^{2r+1} + (3r^2 + 2r)p^{2r+2} \\
&\quad + (2r^2 + 3r + 1)p^{3r} - (4r^2 + r)p^{3r+1} + (2r^2 - 2r)p^{3r+2})/(1-p^r)^4.
\end{aligned}$$

The error is less than 0.0002 for $p = 0.5$ and $n = 12$, for all $r = 1, 2, \dots, 8$. For $p = 0.7$ and $n = 12$, the error is less than 0.008 for all these values of r . The upshot of the analysis is that in the range of typical interest, say $p \leq 0.8, 1 \leq r \leq 8, n \leq 20$, the error in taking the variance divided by v_r^2 to be $nf_r(p) + g_r(p)$ is negligible, being at worst 0.025 but usually much smaller than that.

It should be noted that with current computational capabilities, it is possible to consider the outcomes of all possible rounds of length n so long as $n \leq 25$. For example, playing the 6-th strategy with $n = 18$ leads to payoffs $v_6, 2v_6, 3v_6$ being accrued in proportions

$$\begin{aligned}
\lambda_1 &= -30p^{14} + 78p^{13} - 49p^{12} - 12p^7 + 13p^6 \\
\lambda_2 &= -p^{18} + 15p^{14} - 42p^{13} + 28p^{12} \\
\lambda_3 &= p^{18}
\end{aligned}$$

respectively, of possible rounds. For example, for typical $p = 0.6$, these are approximately 24.2%, 1.8%, and 0.0% respectively, meaning that in 74.0% of rounds, nothing would be banked.

3.3 End Effects Adjustments: In practice, just before time runs out, a sensible player will obviously bank rather than continue trying for a run of r correct answers. This introduces some end effects. To study these effects, we first consider the case of strategy $r = 2$ and use sequences consisting of zeros and ones corresponding to either incorrect and correct answers. For example, the sequence 00011 represents three incorrect answers followed by two correct answers. With this understanding, an extra v_1 will be accrued in a round if the round ends in 01, 0111 or 011111 and so on. If the probability of a correct answer is p as usual, then the expected extra amount accrued in a round is

$$v_1(p(1-p) + p^3(1-p) + p^5(1-p) + \dots + p^k(1-p))$$

where k is the largest odd integer less than or equal to n . For large round length n , this is approximately given by $v_1 p / (1 + p)$.

Now, for strategy $r = 3$, an extra v_1 is accrued in a round if the round ends in 01, 01111 or 01111111 and so on whereas an extra v_2 is accrued if the round ends 011, 011111 or 011111111 and so on. Thus the expected extra amount accrued is

$$v_1(p(1-p) + p^4(1-p) + \dots) + v_2(p^2(1-p) + p^5(1-p) + \dots)$$

This is approximately $(pv_1 + p^2v_2)(1-p)/(1-p^3)$ for large round length.

For general r , for large round length approximately

$$(pv_1 + p^2v_2 + \dots + p^{r-1}v_{r-1})(1-p)/(1-p^r)$$

extra is accrued. We obtain an exact formula for a given n by summing the finite geometric series. By brute force computation, for $r = 6$ and round length $n = 18$, the return and corresponding variance, as a function of p can be obtained in closed form; see Section 4. In this case, the variance turns out to be a polynomial of degree 36. In the general case for given n , while the variance formula becomes rather unwieldy, it is noted that it can still be obtained in closed form.

3.4 Mixed Strategies: We consider here the effect of mixing the r_1 -th and r_2 -th strategies. This can mean mixing within a round or mixing between rounds. The latter is completely understood. If the r_i th strategy has mean μ_i and variance σ_i^2 over each round and we apply the r_1 -th strategy a fraction $1 - \lambda$ of the time and the r_2 -th strategy a fraction λ of the time, then the overall expected return per round is:

$$\mu = (1 - \lambda)\mu_1 + \lambda\mu_2$$

and the overall variance per round is

$$\sigma^2 = (1 - \lambda)\sigma_1^2 + \lambda\sigma_2^2$$

Eliminating λ , we obtain $\sigma^2 = A\mu + B$ with

$$A = (\sigma_2^2 - \sigma_1^2)/(\mu_2 - \mu_1); \quad B = (\mu_2\sigma_1^2 - \mu_1\sigma_2^2)/(\mu_2 - \mu_1)$$

or equivalently $\mu = (\sigma^2 - B)/A$, part of a parabola. To interpret these formulae in the context of the efficiency frontier, we assume $\sigma_1 < \sigma_2$ and first consider the case when $\mu_1 < \mu_2$. Then $A > 0$ and so the arc bows below the line joining (σ_1, μ_1) to (σ_2, μ_2) . If $\mu_1 > \mu_2$, then $A < 0$ and so the arc bows above the line joining (σ_1, μ_1) to (σ_2, μ_2) .

It would be more realistic to perform mixing within a round (for example, if alternating players decided to play different pure strategies). In this case, just as with end effects, formulae can become unwieldy. Simulations indicate that the frequency with which the mixing takes place matters. By this we mean that two strategies could for instance be used equally often but in one case every player in turn switches strategy and in the other case every second player switches strategy. There is, however, no clear trend in simulations to prefer one frequency to another.

4. Some Illustrative Efficiency Frontiers

In this section, we provide some illustrative efficiency frontiers. To this end, we consider the US prime time version of the show with

$$v = [1000 \ 2500 \ 5000 \ 10000 \ 25000 \ 50000 \ 75000 \ 125000]$$

and round length $n = 18$. As discussed in Section 3, taking end effects into account, we obtain explicit polynomials for risk and return which are used to generate efficiency frontiers.

For example, for the 6-th strategy the return and variance as polynomials in p are respectively

$$\begin{aligned} \mu = & 25000p^{18} + 15000p^{17} + 5000p^{16} + 2500p^{15} + 1500p^{14} \\ & -299000p^{13} + 325000p^{12} + 15000p^{11} + 5000p^{10} + 2500p^9 \\ & +1500p^8 - 599000p^7 + 625000p^6 + 15000p^5 + 5000p^4 + 2500p^3 + 1500p^2 + 1000p \end{aligned}$$

Figure 2: **Risk-Return Plot With $p = 0.4$**

$$\begin{aligned}
\sigma^2 \times 10^{-4} = & -62500p^{36} - 75000p^{35} - 47500p^{34} - 27500p^{33} - 17500p^{32} \\
& + 1488000p^{31} - 730125p^{30} - 751750p^{29} - 245725p^{28} - 115300p^{27} - 9065100p^{26} \\
& + 22416000p^{25} - 10997750p^{24} - 2028500p^{23} - 593950p^{22} - 263100p^{21} \\
& - 36142700p^{20} + 76284000p^{19} - 37552875p^{18} - 1502750p^{17} - 389675p^{16} \\
& - 154025p^{15} - 28574775p^{14} + 53916100p^{13} - 24385250p^{12} - 426000p^{11} \\
& - 116450p^{10} - 46225p^9 - 22175p^8 - 3011900p^7 + 3182375p^6 + 50750p^5 \\
& + 6775p^4 + 1575p^3 + 425p^2 + 100p
\end{aligned}$$

In Figures 2-4, the risk-return plots are given for three cases of practical interest.

We now draw the reader's attention to various features of these plots. First, we note that each of the plots is a piecewise quadratic. That is, as discussed in Section 3, when we allow mixing of strategies between rounds, we obtain a risk-return plot which is quadratically parameterized in the coefficient λ which describes the proportion of each of the two constituent strategies used.

Figure 3: **Risk-Return Plot With $p = 0.6$**

In the case of Figure 2, when $p = 0.4$, we see that $r = 1$ is the only efficient point on the plot. That is, for any other strategy including mixtures, a higher risk and a lower return results. Hence, for this case, the efficiency frontier consists of a single point. This implies that in this low probability case, our conclusions are identical with those in the earlier literature. Namely, all strategies except $r = 1$ should be eliminated from consideration.

When $p = 0.6$, we see in Figure 3 that a rather different situation emerges. Now, we have number of efficient points. To summarize, $r = 1$ is an efficient point and points approximately corresponding to $1 < r \leq 4$ are inefficient. Then we see efficiency for the approximate range $4 < r \leq 6$ and inefficiency for $r > 6$. This case is interesting to contrast with the existing literature: While previous authors would regard all strategies other than $r = 6$, the return maximizer, as being sub-optimal, we obtain a rather different conclusion. Depending on a player's aversion to risk, there are a whole host of strategies which are arguably efficient.

Figure 4: **Risk-Return Plot With $p = 0.8$**

Finally, we consider the case $p = 0.8$; see Figure 4. Now it is apparent that all strategies are efficient. In contrast, a return maximizer, playing in accordance with earlier literature, would rule out all strategies other than $r = 8$.

To complete the analysis of this game with $n = 18$, a characterization of all efficient strategies, as a function of p , is given in Figure 5. For each of the eight strategies the set of p values for which it is efficient is indicated in the figure. For example, for $p = 0.6$, only the first, fifth and sixth strategies are efficient while for $p = 0.2$, only the first, third, fourth, seventh and eighth strategies are efficient.

5. Refinements

In this section, we consider two issues that affect the practical player. First, we explore the

Figure 5: Efficiency Characterization for $n = 18$

consequences of picking one point on the efficiency frontier over another. Second, related to this is the question of how close to normally distributed the returns are. The use of efficiency frontier methods, where mean and variance are all that matter, is predicated on subtleties of the distribution being of no consequence.

Consider what happens in a U.S. round of length $n = 18$ if players are answering correctly at the realistic rate of $p = 0.6$. Almost every paper written so far on the game recommends that the players go with the 6-th strategy because of that providing the highest expected return. We saw, however, in the previous section that the 1-st, 5-th, and 6-th strategies are all efficient. An individual might pick any of these, depending on their utility function.

Let us explore, however, the likely consequences of choosing say the 6-th strategy over the others, as a risk-taker (or someone simply following earlier papers) might do. They should know what they are facing. For an expected return per round of \$16,347 versus \$10,800 and

\$14,395 for the 1-st and 5-th strategies respectively, the players are taking quite a risk. For instance, at the end of section 2.2, it was noted that. ignoring end effects, in about 74% of rounds nothing would be banked.

Let us include end effects and consider a realistic show with the number of questions in a round being 18, 17, 15, 14, 13, 12, 11 with the last round counting double (and the probability of a correct answer still 0.6). Lengthy computations using exact formulae show that the probability of the winner leaving with less than \$10,000, \$20,000, \$30,000, \$40,000, \$50,000 is respectively 5.9%, 13.0%, 16.3%, 18.8%, 19.8%. This is all accrued in end effects, and indeed less than \$75,000 is accrued in 31.9% of shows.

On the other hand, the chance of players using the 1-st strategy accruing this much is only about 6%. They expect, however, to win about $111 \times 0.6 \times \$1,000 = \$66,600$, and have almost zero risk of leaving with less than \$50,000.

For the $r = 1$ strategy, the returns follow a binomial distribution, known to approximate the normal distribution for large n . Working with a fixed round length n going to infinity, it appears but is not proven that the limiting distribution of returns is normal. If, however, we count returns as a function of number of attempts (where an attempt ends with a wrong answer or a bank) instead of as a function of round length, then the Central Limit Theorem applies, leading to a limiting normal distribution.

Certainly, for n small relative to r , such as our $n = 18, r = 6$ case where the distribution of returns was computed at the end of Section 2.2, it does not look at all like a normal distribution. Although this casts doubt on the use solely of mean and variance to study the distribution, detailed examples such as that considered above show that the efficiency frontier approach still works well in practice. For $n > 25$, while exact mean and variance formulae are obtainable in principle, excessive computation would be required. Hence, to see whether normality “kicks in” for larger values of n , we carried out a Monte Carlo simulation. A typical result is given in Figure 5. With $n = 30$, $p = 0.62$, $r = 3$ and 20,000 trials, the plot suggests

Figure 6: **Distribution of Payoffs for $p = 0.62, r = 3, n = 30$ and 20,000 Trials**

that asymptotic normality is the case. That is, the normal distribution fit to the histogram appears quite reasonable.

6. Conclusion

In this paper, our primary objective was to describe a new framework for analysis of the Weak-est Link — one which not only accounts for return but also includes risk considerations. As a secondary objective, our goal was to refine previous analyses to include realistic factors such as end effects and mixing of strategies. Consistent with the economic literature, our focal point was the so-called efficiency frontier; i.e., without detailed knowledge of an individual player's utility function, our point of view is that no risk-return pair on the frontier is preferred to another.

Our most important finding is the following: In contrast to previous authors, we see that there are many more strategies which are justifiable. That is, certain strategies which are discounted

as being sub-optimal in a return maximization framework become efficient in a risk-adjusted framework. This phenomenon is epitomized by the case considered in Figure 3. While previous authors would eliminate all strategies other than $r = 6$, risk adjustment leads to a whole host of strategies which are arguably justifiable.

References

- [1] K. Antonovics, P. Arcidiacono, and R. Walsh, “Games and Discrimination: Lessons From the Weakest Link,” UCSD Economics Discussion Papers 2003-03, UCSD Economics Department, 2003.
- [2] N. Boston, “Strategies for the Weakest Link,” *American Math Monthly* 110, Number 4, pp. 330–333. 2003.
- [3] G. Christos and J. Simpson, “Optimal Strategy in the Weakest Link.” *Gazette of the Australian Math Society*, vol. 29, pp. 269–272, 2002.
- [4] E. Fama and M. H. Miller, *The Theory of Finance*, Holt, Rinehart and Winston, New York, 1972.
- [5] P. Février and L. Linnemer, “Strengths of the Weakest Link,” Preprint 200224, Institut National de la Statistique et des Etudes Economiques, 2002.
- [6] M. Haan, B. Los, Y. Riyanto, and M. Van Geest, “The Weakest Link. A Field Experiment in Rational Decision Making,” Experimental 0203001, Economics Working Paper Archive at WUSTL, 2002.
- [7] J. Haigh, “The Weakest Link.” *Journal of the Royal Statistical Society : Series D (The Statistician)* vol. 52, no. 2, pp. 219–226, 2003; also in “Taking Chances: Winning with Probability” Oxford University Press, 2003.
- [8] S. Levitt, “Testing Theories of Discrimination: Evidence from Weakest Link.” NBER Working Papers 9449, National Bureau of Economic Research, Inc., 2003.
- [9] P. A. Samuelson, *Economics*, tenth edition, McGraw-Hill Book Company, New York, 1976.
- [10] R. E. Steuer, *Multiple Criteria Optimization: Theory, Computation and Application*. New York: Wiley, 1986.
- [11] L. Thomas, “The Best Banking Strategy When Playing the Weakest Link.” *Journal of the Operations Research Society*, vol. 54, pp. 747–750, 2003.