Intervals of the Muchnik Lattice

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Abstract

Answering a question raised by Terwijn, we give a lattice-theoretic characterization of the intervals of the Muchnik lattice as a subset of the distributive lattices, valid for all intervals satisfying a cardinality constraint on the size of antichains.

1 Introduction

The relation of Turing reducibility answers the following question: given two problems each with a unique solution, which we may think of as elements \( A, B \in \omega^\omega \), is there a computable way of finding the solution \( A \) to the first problem from the solution \( B \) to the second problem? If so, we say that \( A \leq_T B \). This reducibility yields a natural equivalence relation, whose equivalence classes are the Turing degrees. The induced partial order of Turing degrees has many interesting and unexpected properties, and it would be an understatement to say that it has been the object of a lot of study.

If, instead, our problems may have any number of solutions, we may think of a problem as its set of solutions \( S \subseteq \omega^\omega \). Then we may ask: given two such problems \( S \) and \( T \), is there a computable way of finding a solution to the first problem (an element of \( S \)) given a solution to the second (an element of \( T \))? There are two possible ways to answer this question: if we require that there is a uniformly computable way to compute, given an element of \( T \), some element of \( S \) (which may depend on the element of \( T \) we are given) then we obtain Medvedev, or strong, reducibility, denoted \( S \leq_M T \). If we require only only nonuniform reducibility – that is, that for any element of \( T \), there is some element of \( S \) which is Turing reducible to it – we obtain Muchnik, or weak, reducibility, denoted \( S \leq_w T \). Medvedev and Muchnik reducibilities yield respectively the Medvedev and Muchnik degrees, and the respective partial orders on those degrees are the Medvedev and Muchnik lattices. These can be regarded as generalizations of the Turing degrees, and while they are not as universally studied, they are interesting in their own right.

Terwijn has studied the structures of these lattices, in particular, he has shown [7] that the finite intervals of the Medvedev lattice are exactly the finite Boolean algebras, and the infinite intervals all have antichains of cardinality \( 2^{2^{\aleph_0}} \). Additionally, he has given [8] a characterization of the finite intervals of the Muchnik lattice, which admit a lot more variety, and tantalizingly noted that the situation is more complicated for infinite intervals; in particular, that some countable linear orders (but not others) are intervals of the Muchnik lattice, and that there are intervals with width \( \aleph_0 \), intervals with antichains of size \( 2^{\aleph_0} \) but not \( 2^{2^{\aleph_0}} \), and intervals (in particular the whole lattice) with antichains of size \( 2^{2^{\aleph_0}} \).

Inspired by these hints, we will give a lattice-theoretic characterization that encompasses Terwijn’s characterization of the finite intervals of the Muchnik lattice and also applies to many infinite intervals. In particular, we can characterize all such intervals with no antichains of cardinality \( 2^{\aleph_2} \).

An overview of mass problems and Muchnik reducibility can be found in Rogers [6], while Grätzer [4] and Davey and Priestley [3] contain background in lattice theory.

2 The Muchnik Degrees

We begin by recalling the definitions of mass problems and Muchnik reducibility of these mass problems.

Definition 2.1. A mass problem is a collection \( S \subseteq \omega^\omega \).
**Definition 2.2.** Let $S$ and $T$ be mass problems. Then we say $S$ is *Muchnik reducible* (or *weakly reducible*) to $T$, written $S \leq_w T$, if for every $B \in T$, there is some $A \in S$ such that $A \leq_T B$ (that is, $A$ is Turing reducible to $B$).

Muchnik reducibility yields a notion of equivalence of mass problems.

**Definition 2.3.** We say that $S$ is *Muchnik equivalent* to $T$ (written $S \equiv_w T$) if and only if $S \leq_w T$ and $T \leq_w S$.

It is not difficult to see that Muchnik reducibility is transitive, and thus that Muchnik equivalence does indeed give an equivalence relation on mass problems. Thus, we can define the *Muchnik degrees* and their ordering:

**Definition 2.4.** A *Muchnik degree* is a Muchnik equivalence class of mass problems. The Muchnik degrees are ordered by $\leq_w$ in the natural way.

It is well-known that the Muchnik degrees, together with their ordering, form a distributive lattice. It is the goal of this paper to describe, as much as possible, the intervals of this lattice. To do this, we will first need to develop an easier way of working with Muchnik degrees. In particular, we would like to have something of a canonical representative for each Muchnik degree, and an easy way of delving into their internal structures and comparing them. Most of this simplifying work is folklore, but it is worth making explicit, because by the end our way of looking at Muchnik degrees will be much different than the above definition.

**Lemma 2.5.** The Muchnik degree of a mass problem depends only on the Turing degrees of its members. As a consequence, Muchnik reducibility has the same property.

*Proof.* Suppose that the members of the mass problems $S$ and $T$ yield the same sets of Turing degrees. Then, take any $A \in T$. There is some $B \in S$ such that $B \equiv_T A$. In particular, $B \leq_T A$. Thus, $S \leq_w T$. By the same argument, $T \leq_w S$. Hence $S$ and $T$ have the same Muchnik degree.

In light of Lemma 2.5, we can stop thinking about Muchnik degrees as collections of mass problems, and instead as collections of sets of Turing degrees. It is worth noting that this does not work in the case of Medvedev reducibility. Now, we would like to pick a canonical member of a Muchnik degree.

**Lemma 2.6.** If $S$ is a collection of Turing degrees, then it is Muchnik equivalent to $\text{ucl}(S)$, where $\text{ucl}()$ denotes the upward closure in the set of Turing degrees; that is, $\text{ucl}(S)$ is the set of all Turing degrees in upper cones of elements of $S$.

*Proof.* It is certain that $\text{ucl}(S) \leq_w S$, since every element of $S$ is also an element of $\text{ucl}(S)$, and computes itself. On the other hand, every $a \in \text{ucl}(S)$ is in the upper cone of some $b \in S$, and we have $b \leq_T a$. Hence $S \leq_w \text{ucl}(S)$ also.

So we can restrict our attention to upward closed sets of Turing degrees. Does this give us our canonical representative? It turns out that it does.

**Lemma 2.7.** Let $S$ and $T$ be different upward closed sets of Turing degrees. Then they are not Muchnik equivalent.

*Proof.* Since $S$ and $T$ are different, it follows that one of them (say $S$) contains a Turing degree $a$ that is not in the other. Moreover, since $T$ is upward closed, $a$ cannot be in the upper cone of any Turing degree in $T$ (otherwise $T$ would also contain $a$). Hence there is no $b \in T$ such that $b \leq_T a$, and hence $T \not\leq_w S$. Thus they are not equivalent.

Since we now have a canonical representative for the Muchnik degrees, we would like to be able to give a nice characterization of the ordering in terms of those canonical representatives. We can.

**Lemma 2.8.** The ordering on the Muchnik degrees is given by reverse inclusion of their representatives. That is, if $S$ and $T$ are upward closed sets of Turing degrees, then $S \leq_w T$ if and only if $T \subseteq S$.
3 Some Lattice Theory

We also need a number of ideas from lattice theory.

**Definition 3.1.** A lattice $L$ is complete if, for every subset $S \subseteq L$, $S$ has a supremum in $L$.

If $L$ is a complete lattice, the supremum of a subset $S \subseteq L$ in $L$ is sometimes called the join of $S$ and will be written $\bigvee S$ (or $\vee S$ if the lattice $L$ is clear). In a complete lattice, every subset $S$ also has an infimum (or meet): the supremum of the set of lower bounds of $S$. Every finite lattice is necessarily complete, and every complete lattice has both a least and greatest element.

**Definition 3.2.** By an interval in a lattice $L$ we mean a closed interval: any two (not necessarily different) elements $A \leq B$ of $L$, together with any $X \in L$ such that $A \leq X \leq B$.

**Proposition 3.3.** If $L$ is a complete lattice and $M$ is an interval of $L$, then $M$ is also a complete lattice.

**Proof.** Let $S \subseteq M$ and let $A$ and $B$ be the least and greatest elements of $M$, respectively. If $S$ is empty, then $\bigvee_M S$ is $A$, so suppose that $S$ is nonempty. Then $S \subseteq L$; certainly $\bigvee_L S \leq B$, since $B \in L$ and $B$ is an upper bound for all the elements of $S$. Further, for $X \in S$, we have $A \leq X \leq \bigvee_L S \leq B$, so it follows that $\bigvee_L S \in M$. Since every element of $M$ is also in $L$, $\bigvee_L S$ must be the supremum of $S$ in $M$ as well.

For a general partial order, we have the notion of a convex subset.

**Definition 3.4.** Let $P$ be a partial order. A subset $S \subseteq P$ is convex if for every $x, y \in S$ and $a \in P$, $x \leq a \leq y$ implies that $a \in S$.

Observe that, unlike intervals of lattices, convex subsets of partial orders need not have endpoints.

Filling a similar role to the join-irreducible elements in Terwijn’s characterization in [8] of the finite intervals of $\mathcal{M}_w$, we have the completely join-prime elements and the suborder they form.

**Definition 3.5.** Let $L$ be a complete lattice. An element $X$ is called completely join-prime if for any $S \subseteq L$, $X \leq \bigvee S$ implies that there is some $Y \in S$ such that $X \leq Y$.
**Definition 3.6.** Let $L$ be a complete lattice. The partial order of completely join-prime elements of $L$ is $\mathcal{J}_P(L)$.

The completely join-prime elements of the Muchnik lattice are particularly nice:

**Lemma 3.7.** The completely join-prime elements of $\mathcal{M}_w$ are the downward closures of single Turing degrees. Therefore, $\mathcal{J}_P(\mathcal{M}_w) \cong \mathcal{P}$.

**Proof.** First we show that the downward closure $A$ of a Turing degree $a$ is completely join-prime. Let $S \subseteq \mathcal{M}_w$ and suppose that $A \leq \bigvee S$. Then $a \in X$ for some $X \in S$, and hence $A \leq X$. This shows that $A$ is completely join-prime.

Second, suppose that $A \in \mathcal{M}_w$ is completely join-prime. Let $S = \{dcl(x) \mid x \in A\}$. Then $A \leq \bigvee S$, so it follows that $A \leq dcl(a)$ for some $a \in A$. Since $dcl(a) \leq A$ for any $a \in A$, it follows that $A$ must be the downward closure of the single Turing degree $a$. \qed

We also have a way of extracting a lattice from a partial order.

**Definition 3.8.** Let $P$ be a partial order. Then $\mathcal{O}(P)$ is the lattice of downward closed subsets of $P$, ordered by inclusion.

Terwijn [8] calls this $H(P)$, but we will follow Davey and Priestley [3] in calling it $\mathcal{O}(P)$. In light of Definition 3.8, we can restate Lemma 2.9 as $\mathcal{M}_w \cong \mathcal{O}(\mathcal{P})$.

Terwijn’s characterization of the finite intervals in $\mathcal{M}_w$ rests on the duality between a finite distributive lattice $L$ and the partially ordered set $\mathcal{J}(L)$ of its join-irreducible elements, with $L \cong \mathcal{O}(\mathcal{J}(L))$. This duality does not hold for all infinite distributive lattices, but for a particular class of lattices, which happens to include the intervals of $\mathcal{M}_w$, we can replace $\mathcal{J}(L)$ with $\mathcal{J}_P(L)$ and still get this duality.

**Definition 3.9.** A complete lattice $L$ is completely distributive if it satisfies, for every doubly indexed subset $\{x_{ij}\}_{i \in I, j \in J}$ of $L$:

$$\bigwedge_{i \in I} \left( \bigvee_{j \in J} x_{ij} \right) = \bigvee_{\alpha : I \to J} \left( \bigwedge_{i \in I} x_{i\alpha(i)} \right).$$

In particular, meets distribute over arbitrary joins and joins distribute over arbitrary meets.

**Definition 3.10.** An element $k$ of a complete lattice $L$ is compact if, for every subset $T \subseteq L$, $k \leq \bigvee T$ implies that there is some finite $T' \subseteq S$ such that $k \leq \bigvee T'$. A complete lattice $L$ is algebraic if, for each $a \in L$, $a = \bigvee \{ k \in L \mid k \text{ compact and } k \leq a \}$.

These definitions, and discussion of them, can be found in [3].

**Definition 3.11.** A lattice is superalgebraic if it is algebraic and completely distributive.

As it turns out, there are several equivalent conditions for a lattice being superalgebraic, and it is these equivalent conditions which we are interested in.

**Theorem 3.12** ([3], 10.29 and [2], 2.5). Let $L$ be a lattice. Then the following are equivalent:

- $L$ is superalgebraic.
- $L \cong \mathcal{O}(P)$ for some partially ordered set $P$.
- Every element of $L$ is the join of a set of completely join-prime elements of $L$.
- (Duality) $L \cong \mathcal{O}(\mathcal{J}_P(L))$ via the isomorphism $a \mapsto \{ x \in \mathcal{J}_P(L) \mid x \leq a \}$.

**Corollary 3.13.** $\mathcal{M}_w$ is a superalgebraic lattice.

**Proof.** By Lemma 2.9 we know that $\mathcal{M}_w \cong \mathcal{O}(\mathcal{P})$. \qed

Just as distributivity and completeness are preserved by taking intervals, so too is the property of being superalgebraic.

Proof. Let \( L \) be a superalgebraic lattice and \( M \) be an interval of \( L \), and let the least element of \( M \) be \( A \). If \( X \vee A \) is an element of \( M \) and \( X \) is completely join-prime in \( L \), then either \( X \vee A \) is completely join-prime in \( M \) or \( X \vee A = A \). For suppose that \( S \subseteq M \) and \( X \vee A \leq \bigvee S \). If \( S \) is empty, then \( X \vee A = A \); otherwise, \( \bigvee S \) is the same in both \( M \) and \( L \). In that case, certainly \( S \subseteq L \), so by the fact that \( X \) is completely join-prime in \( L \), we see that for some \( Y \in S \), \( X \leq Y \), and hence \( X \vee A \leq Y \vee A = Y \), with the last equality because \( A \leq Y \). This implies that \( X \vee A \) is completely join-prime in \( M \).

Now let \( X \) be any element of \( M \). If \( X \) is the least element of \( M \), it is the supremum of the empty set and we are done. Otherwise, let \( A \) be the least element of \( M \). Since \( L \) is superalgebraic, there is some set \( S \) of completely join-prime elements of \( L \) such that \( X = \bigvee S \); the set is nonempty since \( X \) is not the least element of \( M \) and thus not the least element of \( L \); indeed, \( S \) must contain some element \( Y \leq A \). For every element \( Y \in S \), observe that \( A \leq Y \vee A \leq X \vee A = X \), so that \( Y \vee A \in M \). Letting \( S_A = \{ Y \vee A \mid Y \in S \} \), we see by the preceding paragraph that every element of \( S_A \) is either the least element \( A \) of \( M \) or a completely join-prime element of \( M \), and that there is at least one of the latter. Letting \( T = S_A \setminus A \), we have that \( \bigvee T = \bigvee S_A = A \vee (\bigvee S) = A \vee X = X \), so that \( X \) is the join of a set of completely join-prime elements of \( M \). It follows that \( M \) is superalgebraic.

Taking intervals also has a nice relationship with the duality condition.

Lemma 3.15. Let \( L \) and \( M \) be superalgebraic lattices. Then \( M \) is isomorphic to an interval of \( L \) if and only if \( J_P(M) \) is isomorphic to a convex subset of \( J_P(L) \).

Proof. First let \( M \) be an interval of \( L \), and let \( A \) and \( B \) be the least and greatest elements of \( M \), respectively. Let \( R \) be the set of completely join-prime elements \( X \in L \) such that \( X \leq A \) and \( X \leq B \). \( R \) is clearly a convex subset of \( J_P(L) \). By the proof of Lemma 3.14 we know that for such \( X \), \( X \vee A \) is completely join-prime in \( M \). Conversely, if \( Y \) is completely join-prime in \( M \), then there is (by complete distributivity) a least \( X \in L \) such that \( X \vee A = Y \). This \( X \) is completely join-prime in \( L \). For if not, take a set \( S \subseteq L \) with \( X \leq \bigvee S \) and \( X \not\leq Z \) for any \( Z \in S \). Then \( S' = \{ (Z \land B) \lor A \mid Z \in S \} \) is a subset of \( M \) with the same property for \( Y \), contradicting that \( Y \) is completely join-prime in \( M \). It follows that \( X \in R \). Finally, if \( X_1 \) and \( X_2 \) are two elements of \( R \) (without loss of generality, \( X_2 \leq X_1 \)), then \( X_1 \lor A = X_2 \lor A \) implies that \( X_2 \leq X_1 \lor A \) and thus that \( X_2 \) is not join-prime, a contradiction. Hence we have an isomorphism between \( R \) and \( J_P(M) \).

Second, suppose that \( J_P(M) \) is isomorphic to a convex subset of \( J_P(L) \). We will show that every such convex subset is of the form above: it is the set of elements \( X \in J_P(L) \) such that \( X \leq A \) and \( X \leq B \) for some \( A \leq B \). To see this, let \( R \) be a convex subset of \( J_P(L) \). Let \( R_L \) be the set of elements of \( J_P(L) \) contained in the downward closure of \( R \) but not contained in \( R \). Set \( A = \bigvee R_L \) and \( B = \bigvee R \). Certainly we have \( X \leq B \) for every \( X \in R \). Additionally, if \( X \in R \), then \( X \leq A \), since \( X \not\leq Y \) for any \( Y \in R_L \) (\( Y \) is in the downward closure of \( R \), and if some element of \( R \) were below \( Y \), then by convexity of \( R \), \( Y \) would be in \( R \)). On the other hand, if \( X \in J_P(L) \) and \( X \leq B = \bigvee R \), then \( X \) lies below some element of \( R \) (and thus it lies in the downward closure of \( R \)) because \( X \) is completely join-prime, and if \( X \leq A \), then it does not lie in \( R_L \). So any \( X \in J_P(L) \) with \( X \not\leq A \) and \( X \leq B \) must be in \( R \). Now we observe that the first direction shows that when \( R \) is of this form, it is isomorphic to \( J_P(T) \) for the interval \( T \) in \( L \) with least element \( A \) and greatest element \( B \). By the duality condition for superalgebraic lattices and the fact that \( T \) is superalgebraic (by Lemma 3.14), we have \( T \cong O(J_P(T)) \cong O(J_P(M)) \cong M \); that is, \( M \) is isomorphic to an interval of \( L \).

4 The Main Theorem

We are now ready to state our characterization of intervals in the Muchnik lattice.

Theorem 4.1 (Main Theorem). A lattice \( L \) with no antichains of cardinality \( 2^{2\omega} \) is isomorphic to an interval of the Muchnik lattice \( \mathcal{M}_a \) if and only if the following hold:

1. \( L \) is superalgebraic,
2. \( J_P(L) \) is an initial segment of an upper semilattice, and
3. \( J_P(L) \) has the countable predecessor property.
Proof that the conditions are necessary. Let $L$ be an interval of the Muchnik lattice $M_c$. By Corollary 3.13, $M_a$ is superalgebraic. By Lemma 3.14, $L$ is also superalgebraic. By Lemma 3.7, $\mathcal{J}_P(M_a) \cong \mathcal{D}$. By Lemma 3.15, it follows that $\mathcal{J}_P(L)$ is isomorphic to a convex subset of the Turing degrees $\mathcal{D}$. Because $\mathcal{D}$ is an upper semilattice, every convex subset $\mathcal{D}$ is an initial segment of an upper semilattice, and because $\mathcal{D}$ has the countable predecessor property, so does every convex subset of $\mathcal{D}$. Hence $\mathcal{J}_P(L)$ has these properties as well.

The proof that these conditions are sufficient is a little more involved.

Proof that the conditions are sufficient. Let $L$ be a superalgebraic lattice such that $\mathcal{J}_P(L)$ is the initial segment of an upper semilattice and has the countable predecessor property.

First, we need a minor set-theoretic lemma.

Lemma 4.2. For any infinite cardinal $\kappa$, the powerset $\mathcal{P}(\kappa)$ has an antichain of cardinality $2^\kappa$.

Proof. For any subset $X \subseteq \kappa$, define $A_X \subset \kappa + \kappa$ (where the summation is ordinal arithmetic) by $\alpha \in A_X$ if and only if $\alpha \in X$, and $\kappa + \alpha \in A_X$ if and only if $\alpha \notin X$, where $\alpha$ ranges over ordinals less than $\kappa$. Then if $X \neq Y$, $A_X$ and $A_Y$ are incomparable. It follows that $\mathcal{P}(\kappa + \kappa)$ has an antichain of cardinality $2^\kappa$. But taking a bijection $f : \kappa + \kappa \to \kappa$ (which exists because $\kappa$ is infinite) we get that $\mathcal{P}(\kappa) \cong \mathcal{P}(\kappa + \kappa)$, so that $\mathcal{P}(\kappa)$ must have an antichain of cardinality $2^\kappa$.

We can use this to put an upper bound on the size of antichains in $\mathcal{J}_P(L)$.

Lemma 4.3. $\mathcal{J}_P(L)$ has no antichains of cardinality $\aleph_2$.

Proof. Suppose that $\{A_n\}_{n \in \omega_2}$ was an antichain in $\mathcal{J}_P(L)$. Let $\{S_n\}_{n < 2^{\omega_2}} \subset \mathcal{P}(\omega_2)$ be an antichain (under $\subseteq$) of subsets of $\omega_2$; such an antichain exists by Lemma 4.2. Define $X_\alpha = \sup\{A_n\}_{n \in S_\alpha}$. We will show that $\{X_\alpha\}_{\alpha < 2^{\omega_2}}$ is an antichain in $L$.

Let $\alpha \neq \beta$. Suppose for contradiction that $X_\alpha \subseteq X_\beta$ (the other ordering works the same way). Then there is some $p \in S_\alpha \setminus S_\beta$, and $A_p \subseteq X_\beta \setminus X_\alpha$. Since $A_p$ is completely join-prime, there is some $q \in S_\beta$ such that $A_p \subseteq A_q$. But $p \neq q$, since $p \not\in S_\beta$, and thus this is a contradiction, since the $A_n$ were incomparable. Thus $\{X_\alpha\}_{\alpha < 2^{\omega_2}}$ is an antichain in $L$.

But $L$ has no antichains of cardinality $2^{\aleph_2}$, so this is a contradiction. Hence $\mathcal{J}_P(L)$ has no antichains of cardinality $\aleph_2$.

Lemma 4.4. Suppose that $P$ is a partial order with no antichains of cardinality $\aleph_2$ and the countable predecessor property. Then $P$ has cardinality at most $\aleph_1$.

Proof. We will show that $P$ is the union of a chain of length at most $\omega_1$ of subsets of $P$ which have cardinality at most $\aleph_1$. Define subsets $P_\alpha$ of $P$ in the following way:

$P_0$: Let $X_0$ be any maximal antichain in $P$. By hypothesis, it has at most $\aleph_1$ elements. Below each element of $X$ there are at most countably many elements of $P$. Hence the downward closure of $X_0$ in $P$ is a union of at most $\aleph_1$ sets, each with cardinality at most $\aleph_0$, hence it has cardinality at most $\aleph_1$. Let $P_0$ be the downward closure of $X_0$.

Successor stages $\alpha + 1$: If $P = P_\alpha$ stop, as $P$ thus has cardinality at most $\aleph_1$. Otherwise, take a maximal antichain $X_{\alpha+1}$ in $P \setminus P_\alpha$. It is an antichain in $P$ so it is of cardinality at most $\aleph_1$; also, it is nonempty. By the countable predecessor property, the downward closure of $X_{\alpha+1}$ in $P$ has cardinality at most $\aleph_1$. Let $P_{\alpha+1}$ be the union of $P_\alpha$ and the downward closure of $X_{\alpha+1}$; it has cardinality at most $\aleph_1$.

Countable limit stages $\alpha$: Define $P_\alpha = \bigcup_{\beta < \alpha} P_\beta$. It is a countable union of sets of cardinality at most $\aleph_1$, so it has cardinality at most $\aleph_1$.

It remains to prove that $P = \bigcup_{\alpha < \omega_1} P_\alpha$. Suppose not. Then there is some $x \in P$ such that $x \not\in \bigcup_{\alpha < \omega_1} P_\alpha$. At every successor stage $\alpha + 1$, $x \not\in P_{\alpha+1}$, so it follows that $x$ is not in $P_\alpha$, nor is it below any element of $X_{\alpha+1}$. By maximality of $X_{\alpha+1}$, it follows that $x$ must be above some element of $X_{\alpha+1}$. It follows that, for each countable $\alpha$, some element of $X_{\alpha+1}$ is a predecessor of $x$. But these are all distinct, and there are uncountably many countable ordinals $\alpha$, so it follows that $x$ has uncountably many predecessors, contradicting the fact that $P$ has the countable predecessor property. Hence there is no such $x$ and $P = \bigcup_{\alpha < \omega_1} P_\alpha$.

This expresses $P$ as a union of a collection of size $\aleph_1$ of sets with cardinality at most $\aleph_1$, so it follows that $P$ is of cardinality at most $\aleph_1$.
Together, these give us an important fact about \( \mathcal{J}_P(L) \).

**Lemma 4.5.** \( \mathcal{J}_P(L) \) has cardinality at most \( \aleph_1 \).

*Proof.* Lemma 4.4 applies to \( \mathcal{J}_P(L) \) by Lemma 4.3.

We can now use a powerful result of Abraham and Shore [1] to map \( \mathcal{J}_P(L) \) to a nice part of \( \mathcal{D} \).

**Proposition 4.6.** Let \( P_0 \) be the result of adding a smallest element \( 0 \) to \( \mathcal{J}_P(L) \). Then \( P_0 \) is isomorphic to an initial segment of the upper semilattice \( \mathcal{D} \).

*Proof.* By assumption, \( \mathcal{J}_P(L) \), and thus \( P_0 \), has the countable predecessor property and is an initial segment of an upper semilattice. By Lemma 4.5, \( P_0 \) has cardinality at most \( \aleph_1 \), and by construction, \( P_0 \) has a least element. Abraham and Shore proved [1, Theorem 3.12] that every such partial order is isomorphic to an initial segment of \( \mathcal{D} \).

It follows by removing the least element of \( P_0 \) that \( \mathcal{J}_P(L) \) is isomorphic to a convex subset of \( \mathcal{D} \). By Lemma 3.7, we have that \( \mathcal{J}_P(L) \) is isomorphic to a convex subset of \( \mathcal{J}P(\mathcal{M}_w) \), and thus by Lemma 3.15, it follows that \( L \) is isomorphic to an interval of \( \mathcal{M}_w \).

As corollaries to Theorem 4.1 we can get interesting characterizations in several special cases:

**Corollary 4.7.** [8, Theorem 3.14] A finite distributive lattice \( L \) is isomorphic to an interval of \( \mathcal{M}_w \) if and only if the join-irreducible elements \( \mathcal{J}(L) \) of \( L \) form an initial segment of an upper semilattice.

*Proof.* Such a lattice is automatically superalgebraic, and for finite lattices the join-irreducible elements are necessarily completely join-prime, so \( \mathcal{J}_P(L) = \mathcal{J}(L) \). Finally, \( \mathcal{J}_P(L) \) is finite and therefore necessarily has the countable predecessor property.

The reader will observe that Terwijn also uses the condition of not being “double-diamond-like” – that is, not having a pair of elements of \( \mathcal{J}(L) \) with two minimal mutual upper bounds – as an equivalent of forming an initial segment of an upper semilattice. In the finite case, these coincide just as the notions of join-irreducible and completely join-prime do. As soon as one steps into the infinite case, both of these simplifications break down. For instance, the linear order \( \omega + 1 \) occurs as an interval of \( \mathcal{M}_w \); its greatest element is join-irreducible but not completely join-prime. Similarly, one has a partial order \( P \) with two minimal elements and an infinite descending chain of mutual upper bounds for these two elements; the lattice \( L \) with \( \mathcal{J}_P(L) = P \) is not double-diamond-like, but neither is \( P \) an initial segment of an upper semilattice, and so this \( L \) does not occur as an interval of \( \mathcal{M}_w \).

**Corollary 4.8.** A linear order \( L \) is isomorphic to an interval of \( \mathcal{M}_w \) if and only if it is complete and the set of successors is dense in \( L \) and has the countable predecessor property.

*Proof.* The successor elements in a linear order are exactly the completely join-prime elements. The set of successors being dense in \( L \) is exactly the second condition in Theorem 3.12, so this just says that \( L \) is superalgebraic. \( \mathcal{J}_P(L) \), the set of successors, is a linear order and thus automatically an initial segment of an upper semilattice; its having the countable predecessor property is just the last condition in Theorem 4.1.

**Corollary 4.9.** A countable lattice \( L \) is isomorphic to an interval of \( \mathcal{M}_w \) if and only if it is superalgebraic and \( \mathcal{J}_P(L) \) is an initial segment of an upper semilattice.

*Proof.* \( \mathcal{J}_P(L) \) is countable and thus automatically has the countable predecessor property.

As a corollary to the *proof* of Theorem 4.1, we also obtain a nice result about initial segments of \( \mathcal{M}_w \).

**Corollary 4.10.** A lattice \( L \) satisfying the conditions in Theorem 4.1 is isomorphic to a closed initial segment of \( \mathcal{M}_w \) if and only if the reduced lattice \( L^- \) obtained by removing the least element of \( L \) is either empty or has a least element.

*Proof.* The case when \( L^- \) is empty corresponds to the interval \([\emptyset, \emptyset]\) in \( \mathcal{M}_w \). Otherwise, the condition that \( L^- \) have a least element is necessary, since, because \( \mathcal{D} \) has a least element \( 0 \), every initial segment of \( \mathcal{M}_w \) has both a least element \((\emptyset)\) and a second-least element \((\{\emptyset\})\). On the other hand, it is sufficient, because applying the proof of the main theorem to \( L^- \) we see that \( L^- \) is in fact isomorphic to an interval of the form \([\emptyset, B]\) for some Muchnik degree \( B \), and hence \( L \) is isomorphic to the initial segment \([\emptyset, B]\).
5 Join-Irreducibles and Intervals with no Uncountable Antichains

The reader will observe that the partial order $\mathcal{J}(L)$ of completely join-prime elements plays in our characterization and proof roughly the role that the partial order $\mathcal{J}(L)$ of join-irreducible elements played in Terwijn’s original paper. Indeed, as we observed in Corollary 4.7, these notions coincide in the finite case, but not more generally. Nevertheless, the join-irreducible elements of $\mathcal{M}_w$ and its intervals do hold some interest.

Definition 5.1. Given a partial order $P$, an ideal of $P$ is a nonempty downward closed subset $S \subseteq P$ such that for every $x, y \in S$, there is some $z \in S$ such that $x, y \leq z$. A principal ideal is the downward closure of a single element of $P$.

We have already observed (Lemma 3.7) that the completely join-prime elements of $\mathcal{M}_w$ are exactly the principal ideals of $\mathcal{P}$; that is, the principal Turing ideals. Since principal ideals and ideals coincide the the ideals of $\mathcal{P}$. This, and something more general, is true.

Theorem 5.2. Let $L$ be a superalgebraic lattice and identify $L$ with $O(\mathcal{J}_P(L))$ via the canonical isomorphism (from Theorem 3.12). Then the join-irreducible elements of $L$ are exactly the ideals of $\mathcal{J}_P(L)$ and the completely join-prime elements of $L$ are exactly the principal ideals of $\mathcal{J}_P(L)$.

Proof. Suppose that $A \subseteq \mathcal{J}_P(L)$ is an ideal, and suppose for contradiction that $A = B \cup C$ for some incomparable $B, C \in O(\mathcal{J}_P(L))$. Then there is some $x \in B$ and $y \in C$ such that $x \notin C$ and $y \notin B$. But since $A$ is an ideal, there is some $z \in A$ such that $x, y \leq z$; this $z$ must be in either $B$ or $C$, which (since $B$ and $C$ are downward closed), implies that both $x$ and $y$ are in one of them. This is a contradiction, so $A$ must be join-irreducible.

Conversely, suppose that $A \subseteq \mathcal{J}_P(L)$ is join-irreducible. Let $x, y \in A$, supposing for contradiction that there is no $z \in A$ such that $x, y \leq z$, and define $B = \{z \in A \mid y \not\leq z\}$ and $C = \{z \in A \mid x \not\leq z\}$. Then $B$ and $C$ are downward closed, $y \notin B$ and $x \notin C$, and $B \cup C = A$ since $x$ and $y$ have no mutual upper bound in $A$. This contradicts $A$ being join-irreducible, so in fact $A$ must be an ideal.

Now consider the principal ideals. Let $A \subseteq \mathcal{J}_P(L)$ be a principal ideal. Then $A = \text{dcl}(x)$ for some $x \in \mathcal{J}_P(L)$; hence $A \subseteq B \in O(\mathcal{J}_P(L))$ if and only if $x \in B$. Let $S \subseteq O(\mathcal{J}_P(L))$ and suppose that $A \subseteq \bigcup S$; then $x \in \bigcup S$ and hence there is some $B \in S$ such that $x \in B$, whence $A \subseteq B$. This shows that $A$ is completely join-prime.

Finally, suppose that $A \subseteq \mathcal{J}_P(L)$ is completely join-prime. Writing $A \subseteq \bigcup_{x \in A} \text{dcl}(x)$ yields that $A \subseteq \text{dcl}(x)$ for some $x \in A$ since $A$ is completely join-prime, and of course $\text{dcl}(x) \subseteq A$. Hence $A$ is a principal ideal. □

As a corollary we have a characterization of the join-irreducible elements of $\mathcal{M}_w$.

Corollary 5.3. The join-irreducible elements of $\mathcal{M}_w$ are exactly the ideals of $\mathcal{P}$.

If we restrict our attention to lattices with no uncountable antichains, it turns out that everything can be expressed nicely in terms of join-irreducibles.

Theorem 5.4. Let $L$ be a superalgebraic lattice with no uncountable antichains. Then every element of $L$ is the join of finitely many join-irreducible elements of $L$.

Observe that if we remove “finitely many” this is true of all superalgebraic lattices by the third condition in Theorem 3.12. On the other hand, with the “finitely many” condition, it is not: consider, for instance, $L = O(P)$ where $P$ consists of uncountably many incomparable elements.

Proof. Continue to identify $L$ with $O(\mathcal{J}_P(L))$. Let $X \in L$ be arbitrary. Because the interval $M = [\emptyset, X]$ is again superalgebraic by Lemma 3.14, every ideal of $\mathcal{J}_P(L)$ contained in $M$ is again an ideal of $\mathcal{J}_P(M)$, and every ideal of $\mathcal{J}_P(M)$ is an ideal of $\mathcal{J}_P(L)$, it suffices to consider the largest element $X \in L$.

Since the union of a chain of ideals is again an ideal, it follows that every ideal of $\mathcal{J}_P(L)$ is contained in a maximal ideal. Since $X$ is the union of (principal) ideals, it is therefore the union of maximal ideals. Our goal will be to get a handle on the maximal ideals of $L$ and show that they are finite in number.
Lemma 5.5. There are at most countably many maximal ideals of \( J_P(L) \).

Proof. Any pair of maximal ideals is necessarily incomparable, so the collection of all maximal ideals is an antichain of \( L \). By assumption \( L \) has no uncountable antichains, so \( J_P(L) \) has at most countably many maximal ideals. \( \square \)

Lemma 5.6. Let \( \mathcal{A} \) and \( \{ \mathcal{B}_n \}_{n \leq N} \) be distinct maximal ideals of \( J_P(L) \) for some \( N \in \omega \). Then \( \mathcal{A} \not\subseteq \bigcup_{n \leq N} \mathcal{B}_n \).

Proof. For every \( n \leq N \), \( \mathcal{A} \not\subseteq \mathcal{B}_n \), and so there is some \( x_n \in \mathcal{A} \) such that \( x_n \not\in \mathcal{B}_n \). Since \( \mathcal{A} \) is an ideal, there is some \( z \in \mathcal{A} \) such that \( x_n \leq z \) for each \( n \leq N \); this \( z \) can therefore not be contained in any \( \mathcal{B}_n \), and hence \( \mathcal{A} \not\subseteq \bigcup_{n \leq N} \mathcal{B}_n \). \( \square \)

Next, we make a somewhat topological definition, broadly inspired by an analogy between maximal ideals and paths in an infinite binary tree.

Definition 5.7. A maximal ideal \( \mathcal{A} \) of a partial order \( P \) is a limit ideal if it is contained in the union of all other maximal ideals of \( P \). Otherwise, \( \mathcal{A} \) is said to be isolated. Equivalently, \( \mathcal{A} \) is isolated if and only if there is some \( x \in \mathcal{A} \) such that \( x \not\in \mathcal{B} \) for any other maximal ideal \( \mathcal{B} \) of \( P \).

Lemma 5.8. If \( \mathcal{A} \subseteq \bigcup_{n \in \omega} \mathcal{B}_n \) is a limit ideal (the union is necessarily countable by Lemma 5.5), then \( \mathcal{A} \subseteq \bigcup_{n \geq N} \mathcal{B}_n \) for any \( N \in \omega \).

Proof. Let \( N \in \omega \) be arbitrary. By Lemma 5.6, there is some \( x \in \mathcal{A} \) which is not contained in any \( \mathcal{B}_n \) for \( n < N \). Let \( y \in \mathcal{A} \) be arbitrary; then because \( \mathcal{A} \) is an ideal there is some \( z_y \in \mathcal{A} \) such that \( x, y \leq z_y \). Since \( x \leq z_y, z_y \not\in \mathcal{B}_n \) for \( n < N \); since \( z_y \in \mathcal{A}, z_y \in \mathcal{B}_n \) for some \( n \), so this must occur for some \( n \geq N \). Since \( \mathcal{B}_n \) is downward closed, \( y \in \mathcal{B}_n \) for some \( n \geq N \), and as this held for every \( y \in \mathcal{A}, \mathcal{A} \subseteq \bigcup_{n \geq N} \mathcal{B}_n \). \( \square \)

We are now ready to prove that \( J_P(L) \) has finitely many maximal ideals. A priori, there are three possible cases:

1. There are finitely many maximal ideals.
2. There are infinitely many isolated ideals.
3. There are finitely many isolated ideals and infinitely many limit ideals.

Our goal is to show that only the first case can hold, so we will prove that each of the other cases is impossible by showing that each leads to a contradiction.

Lemma 5.9. If there are infinitely many isolated ideals, then there is an uncountable antichain in \( L \), a contradiction. So the second case is impossible.

Proof. Each isolated ideal \( \mathcal{A}_n \) contains an element \( x_n \) not in any other isolated ideal; thus, any union of isolated ideals contains that element \( x_n \) if and only if \( \mathcal{A}_n \) is present in the union. Let infinitely many isolated ideals be given by \( \{ \mathcal{A}_n \}_{n \in \omega} \); it follows that \( \bigcup_{n \in S} \mathcal{A}_n \) and \( \bigcup_{n \in T} \mathcal{A}_n \) are incomparable in \( L \) if and only if the subsets \( S \) and \( T \) of \( \omega \) are \( \subseteq \)-incomparable. Since \( 2^\omega \) under \( \subseteq \) has an uncountable antichain (by Lemma 4.2, it follows that there are uncountably many pairwise incomparable unions of isolated ideals, and hence \( L \) has an uncountable antichain, contradicting our hypothesis about \( L \). \( \square \)

Lemma 5.10. If there are finitely many isolated ideals and infinitely many limit ideals, then there are uncountably many limit ideals, contradicting Lemma 5.5. So the third case is impossible.

Proof. Since there are only finitely many isolated ideals, by Lemma 5.8 every limit ideal is contained in the union of other limit ideals. Let \( \{ \mathcal{A}_n \}_{n \in \omega} \) be all of the (countably many) limit ideals, with \( A_i \neq A_j \) for \( i \neq j \). We will construct a limit ideal not on the list by a diagonalization argument, giving a contradiction.

We construct \( \mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n \), with the \( \mathcal{B}_n \) constructed as follows. Let \( b_0 \in \mathcal{A}_0 \) and \( b_0 \) is not contained in any of the finitely many isolated ideals (we can do this by Lemma 5.6), and let \( \mathcal{B}_0 = \text{dcl}(b_0) \). We will then construct by induction \( \mathcal{B}_n \) for \( n \geq 0 \), satisfying the following conditions:

1. \( \mathcal{B}_n \) is a principal ideal.
2. $B_n \subseteq A_{f(n)}$ for some $f(n)$.
3. $B_n \supseteq B_{n-1}$ if $n > 0$.
4. $B_n \nsubseteq A_{n-1}$ if $n > 0$.

Certainly $B_0$ satisfies these conditions with $f(0) = 0$. Now, given $B_n$ satisfying these conditions, we construct $B_{n+1}$. Let $b_n$ be the largest element of $B_n$. Choose $m > n$ such that $b_n \in A_m$; such an $m$ exists by Lemma 5.8 applied to $A_{f(n)}$. Since $m > n$, $A_m \neq A_n$, so there is some $a \in A_m$ such that $a \notin A_n$. Since $A_n$ is an ideal, there is some $b_{n+1} \in A_m$ such that $a, b_n \leq b_{n+1}$. Define $B_{n+1} = \text{cl}(b_{n+1})$. Then $B_{n+1}$ is a principal ideal, $B_{n+1} \subseteq A_{f(n+1)}$ where $f(n+1) = m, b_n \in B_{n+1}$ so that $B_n \subseteq B_{n+1}$, and finally, $B_{n+1} \nsubseteq A_n$, since it contains $a \notin A_n$. Thus by induction we can construct a sequence of $B_n$ satisfying all four conditions.

Finally, define $B = \bigcup_{n \in \omega} B_n$. Since $B$ is the union of a chain of ideals, it is an ideal. By the fourth condition, $B \nsubseteq A_n$. Let $B^*$ be any maximal ideal containing $B$ (this may or may not be $B$ itself). Since $B \nsubseteq A_n$ for any $n$, $B^* \neq A_n$ for any $n$. On the other hand, $B^*$ is not isolated, since it contains $b_0$ which is not an element of any isolated ideal. It must therefore be a new limit ideal which was not on our list, a contradiction. So there are uncountably many limit ideals, which itself contradicts Lemma 5.5.

It therefore follows that there are only finitely many maximal ideals. Since $\mathcal{X}$ is the union (join) of all the maximal ideals, $\mathcal{X}$ is therefore the join of finitely many ideals, which by Theorem 5.2 is the join of finitely many join-irreducibles in $L$.

This applies when $\mathcal{X}$ is the largest element of $L$, but as we observed at the beginning, by passing to the interval $[\emptyset, \mathcal{X}]$ we can obtain the result for an interval in which $\mathcal{X}$ is the largest element and then pull back to the original lattice.

As a corollary we get the following result about intervals in the Muchnik lattice.

**Corollary 5.11.** Let $L$ be an interval of the Muchnik lattice $\mathcal{M}_w$ with no uncountable antichain. Then every element of $L$ is the join of finitely many join-irreducible elements of $L$.

## 6 Intervals with Large Antichains

The characterization of Theorem 4.1 is not quite a complete characterization of the intervals of $\mathcal{M}_w$. There is one pesky condition: that $L$ not have any antichains of size $2^{\aleph_0}$. What about lattices $L$ with larger antichains?

It is certainly possible for $L$ to have antichains larger than that. Indeed, by taking a set $\mathcal{X}$ of infinite size $\kappa$ ($\kappa \leq 2^{\aleph_0}$) of minimal Turing degrees, the interval $L = [\emptyset, \mathcal{X}]$ in $\mathcal{M}_w$ has, by Lemma 4.2, an antichain of size $2^\kappa$. What prevents our using the techniques in this paper to characterize all these intervals of $\mathcal{M}_w$ is exactly the fact that it is unknown which posets of size greater than $\aleph_1$ can be initial segments of the Turing degrees.

Indeed, Groszek and Slaman [5] show that it is consistent with ZFC that $2^{\aleph_0} > \aleph_2$ and that there is a locally finite upper semilattice $P$ of cardinality $\aleph_2$ which cannot be embedded into $\mathcal{D}$. In that case, the lattice $H(P)$ would satisfy all the conditions of Theorem 4.1, except for having an antichain of cardinality $2^{\aleph_2}$, but it would not be isomorphic to an interval of $\mathcal{M}_w$ despite having cardinality smaller than that of $\mathcal{M}_w$. This means that Theorem 4.1 is best possible in the sense that the same characterization does not necessarily apply when we relax the constraint on antichain cardinality.

On the other hand, we can ask how incomplete our characterization is, in the sense of asking which intervals in $\mathcal{M}_w$ it doesn’t catch. This also depends on our model of set theory. For example, if $2^{2^{\aleph_0}} < 2^{\aleph_2}$ (which in particular holds under GCH), our characterization is complete. On the other hand, under Martin’s Axiom + ($2^{\aleph_0} > \aleph_2$), we have $2^{\aleph_2} = 2^{\aleph_0}$, so our characterization fails to tell us about any of the intervals in $\mathcal{M}_w$ with antichains the size of the continuum, which presumably would be nice to know about. We can get around a few (but not all) of these issues if we are willing to restrict the class of lattices $L$ under consideration by the properties of $\mathcal{F}_P(L)$, rather than just by the size of their antichain, thus incorporating some parts of the characterization into the choice of domain. Our definition of $\mathcal{F}_P(L)$ was only for lattices in which every subset of $L$ had a supremum in $L$, but if we let $\mathcal{F}_P(L)$ be empty for other lattices, the following is true:
Theorem 6.1. Let $L$ be a complete lattice such that $\mathcal{J}_\mathcal{P}(L)$ has no antichains of cardinality greater than $\aleph_1$. Then $L$ is isomorphic to an interval of $\mathcal{M}_w$ if and only if $L$ is superalgebraic and $\mathcal{J}_\mathcal{P}(L)$ is an initial segment of an upper semilattice and has the countable predecessor property.

Proof. Exactly the same as the proof of Theorem 4.1, except that Lemma 4.3 is a hypothesis rather than a lemma.

Depending on set theory, this version of the theorem may include more intervals of $\mathcal{M}_w$ than the other. Whether that makes it better is largely a matter of aesthetics. The author feels that restricting the domain based on the cardinality of antichains in the lattice $L$ itself is more natural than doing so based on $\mathcal{J}_\mathcal{P}(L)$, and the characterizations are identical in a wide range of cases including under GCH and under the Proper Forcing Axiom.

References


