1. (9 points) Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be a linear transformation such that 
\[ T(1, 0, 0) = (1, 2, 3), \ T(0, 1, 0) = (2, 1, 4), \ T(0, 0, 1) = (3, 5, 6). \] 
Calculate \( T(4, -1, 2). \)

\[ (4, -1, 2) = 4(1, 0, 0) - 1(0, 1, 0) + 2(0, 0, 1) \] 
so that 
\[ T(4, -1, 2) = 4T(1, 0, 0) - 1T(0, 1, 0) + 2T(0, 0, 1) = 4(1, 2, 3) - 1(2, 1, 4) + 2(3, 5, 6) = (8, 17, 20). \]

2. [8 points] For each of the following pairs \( A, B \) of matrices, determine whether or not \( A \) and \( B \) are similar. Justify your answer in each case.

(a) \( A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \). Yes No. Why?

NO: they have different determinants

(b) \( A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 6 & -4 \\ 1 & 1 \end{bmatrix} \). Yes No. Why?

No: they have different traces

3. (10 points) Let \( V \) and \( W \) be vector spaces of the same dimension \( n \). Let \( v_1, v_2, \ldots, v_n \) be a basis of \( V \). Let \( T : V \to W \) be a bijective linear transformation (isomorphism).

PROVE that \( T(v_1), T(v_2), \ldots, T(v_n) \) is a basis of \( W \).

Suppose that \( c_1T(v_1) + c_2T(v_2) + \ldots + c_nT(v_n) = 0 \). Then using properties of linear transformations we get that 
\[ T(c_1v_1 + c_2v_2 + \ldots + c_nv_n) = T(0) = 0. \] 
Since \( T \) is bijective, this implies that \( c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0 \). Since \( v_1, v_2, \ldots, v_n \) is a basis of \( V \), we conclude that \( c_1, c_2, \ldots, c_n \) all equal 0, and hence \( T(v_1), T(v_2), \ldots, T(v_n) \) are linearly independent. Since \( W \) has dimension \( n \), \( T(v_1), T(v_2), \ldots, T(v_n) \) is a basis of \( W \).

4. [10 points] Let \( A \) and \( B \) be square matrices of order \( n \) with \( A \) similar to \( B \). Prove that \( A^3 \) is similar to \( B^3 \).

We have \( B = PAP^{-1} \) for some nonsingular matrix \( P \). Calculating we get that \( B^3 = PAP^{-1}PAP^{-1}PAP^{-1} = PA^3P^{-1} \). Thus \( A^3 \) is similar to \( B^3 \)

5. (10 points) Consider the standard ordered basis \( \alpha : e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1) \) of \( \mathbb{R}^3 \)
and the ordered basis
\[ \beta : e_2, e_3, e_1 \text{ of } \mathbb{R}^3. \]

Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be the linear transformation given by
\[
T(a, b, c) = (2a - b + c, a + 2b + 4c, 3a + 5c).
\]

Determine \( [T]_{\beta}^\alpha \).

We need to write \( T(e_1), T(e_2), T(e_3) \) as linear combinations of \( e_2, e_3, e_1 \) in that order. Using the given formula for \( T \) we have
\[
T(e_1) = (2, 1, 3) = 1e_2 + 3e_3 + 2e_1
\]
\[
T(e_2) = (-1, 2, 0) = 2e_2 + 0e_3 - 1e_1
\]
\[
T(e_3) = (1, 4, 5) = 4e_2 + 5e_3 + 1e_1.
\]

Hence
\[
[T]_{\beta}^\alpha = \begin{bmatrix}
1 & 2 & 4 \\
3 & 0 & 5 \\
2 & -1 & 1
\end{bmatrix}.
\]

6. (15 points) Let \( \ell \) be the line in the plane through the origin making an angle of \( \theta = \pi/6 \) (30 degrees) with the positive \( x \)-axis. Let \( T \) be the linear transformation on \( \mathbb{R}^2 \) given by the reflection in line \( \ell \). Determine the matrix \( [T]_{\alpha}^\alpha \) of \( T \) with respect to the standard basis \( \alpha : e_1 = (1, 0), e_2 = (0, 1) \) of \( \mathbb{R}^2 \).

We can do \( T \) by rotating by \(-\pi/6\) to bring \( \ell \) to the horizontal axis, reflect about the horizontal axis, and then rotate back by \( \pi/6 \). Thus the matrix is
\[
\begin{bmatrix}
\cos \pi/6 & -\sin \pi/6 \\
\sin \pi/6 & \cos \pi/6
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
\cos -\pi/6 & -\sin -\pi/6 \\
\sin -\pi/6 & \cos -\pi/6
\end{bmatrix}.
\]

It is then an easy matter to substitute for the sines and cosines and carry out the multiplication.

7. [10 points] Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be the linear transformation given by
\[
T(x_1, x_2, x_3) = (x_1 + x_3, x_2 + x_3, 2x_1 + x_3)
\]

Determine, with justification, whether or not

1. \( T \) is injective (one to one),
2. \( T \) is surjective (onto),
3. \( T \) is an isomorphism?
Circle those that are correct and then explain why the statements are correct or not correct.

The matrix of $T$ relative to the standard basis is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix},$$

whose determinant is $-1$. Hence $A$ is nonsingular (invertible) and so $T$ is injective, surjective, and hence an isomorphism.

8. [18 points] Consider $R^4$ and the (standard) Euclidean inner product (the dot product). Answer the following questions (no reason necessary):

(a) Are the vectors $x = (1, 2, -1, 4)$ and $y = (3, 2, 3, -1)$ orthogonal? YES

(b) The length $||x||$ of $X$ equals:

$\sqrt{22}$

(c) If a vector $z$ in $R^4$ is orthogonal to $x$ and $y$, then does $z$ have to be orthogonal to $3x + 2y$? YES

(d) What is the statement of Cauchy-Schwarz inequality for this dot product on $R^4$.

$$|x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4| \leq \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} \sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2}.$$ 

(e) Four nonzero vectors in $R^4$ which are mutually orthogonal must be a basis of $R^4$. YES

(f) Four nonzero vectors of $R^4$ which are a basis must be mutually orthogonal. No