1. [27 points] $A$ and $B$ are real matrices of order 4 with determinants 3, and 7 respectively. Answer the following questions:

1. $\det(-A) = (-1)^4 3 = 3$

2. $\det A^T = 3$

3. $\det B^{-1} = 7^{-1} = 1/7$

4. $\det(AB) = 3 \cdot 7 = 21$

5. If $1 + i$ and $2 - 3i$ are complex eigenvalues of $A$, what are its other two eigenvalues? $1 - i$ and $2 + 3i$.

6. The row space of $A$ equals: $R^4$

7. The product of the eigenvalues of $A^T$ equals $\det A^T = 3$:

8. The product of the eigenvalues of $B^{-1}$ equals $(-1)^4 \det B^{-1} = 1/7$:

9. The dot product of the first column vector of $A$ with the second row vector of $A^{-1}$ equals: 0 (since $A^{-1} \cdot A = I_4$)
2. [14 points] Let \( A = \begin{bmatrix} 8 & 3 \\ -1 & 4 \end{bmatrix} \). Determine an invertible matrix \( Q \) that diagonalizes \( A \): \( Q^{-1}AQ = D \) and the diagonal matrix \( D \).

The characteristic polynomial of \( A \) equals
\[
\lambda^2 - 12\lambda + 35 = (\lambda - 5)(\lambda - 7).
\]
Hence the eigenvalues of \( A \) are 5, 7. We need to find an eigenvector of \( A \) for each of these eigenvalues.

5\(I_2 - A = \begin{bmatrix} -3 & -3 \\ 1 & 1 \end{bmatrix} \).

Hence an eigenvector is \([1 \ -1]^T\).

7\(I_2 - A = \begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix} \).

Hence an eigenvector is \([3 \ -1]^T\) Let

\[ Q = \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix} \]

The \( Q \) is invertible with
\[
Q^{-1} = 1/2 \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix}.
\]

We must have
\[
Q^{-1}AQ = D = \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}.
\]
3. [12 points] Let $A$ and $B$ be orthogonal matrices of order $n$. Let $x$ and $y$ be $n$-tuples written as column vectors.

1. **Prove:**

   $$(Ax) \cdot (Ay) = x \cdot y.$$  

   We have

   $$(Ax) \cdot (Ay) = (Ax)^T(Ay) = x^TA^T Ay = x^TI_n y = x^T y = x \cdot y,$$

   since $A$ is orthogonal and so $A^{-1} = A^T$.

2. **Prove:** $AB$ is an orthogonal matrix.

   We have

   $$(AB)^T = B^T A^T = B^{-1} A^{-1} = (AB)^{-1}.$$  

   Thus $AB$ is an orthogonal matrix.
4. [15 points] Let $A$ and $B$ be 4 by 4 matrices that have the same column space. (Note we are not assuming that $A$ and $B$ have the same columns, only that they have the same column space.)

(i) Are $A$ and $B$ sure to have the same number of pivots?

**YES:** WHY? Because the number of pivots equals the dimension of the column space.

**NO:** Exhibit an Example

(ii) Are $A$ and $B$ sure to have the same row space?

**YES:** WHY?

**NO:** Exhibit an Example.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

have the the same column space but clearly different row spaces. (The first row of $A$ is a basis for the row space of $A$, and the first row of $B$ is a basis for the row space of $B$, and clearly these row spaces are different.)

(iii) If $A$ is invertible, are you sure that $B$ is invertible?

**YES:** WHY? Then the dimension of column space of $A$, and so of $B$ equals 4. So rank of $B$ is 4 and $B$ is invertible too.

**NO:** Exhibit an Example.
5. [20 points]

(i) Find an **orthonormal basis** for the null space $U$ of the equation $x + y + z = 0$.

Clearly a basis of the null space $U$ is $x = [1 -1 0]^T$ and $y = [1 0 -1]^T$. Applying the G-S process, we get

$$u_1 = \frac{1}{\sqrt{2}}[1 -1 0]^T.$$

projecting $y$ onto the space spanned by $u$ and taking the difference vector we get

$$[1 0 -1]^T - \frac{1}{\sqrt{2}}u_1 = [1/2 1/2 -1]^T.$$

Dividing by its length $\sqrt{6}/2$, we get

$$u_2 = \frac{2}{\sqrt{6}}[1/2 1/2 -1]^T = \frac{1}{\sqrt{6}}[1 1; -2]^T.$$

(ii) Find the **orthogonal projection** of the vector $b = [1 2 6]$ onto $U$.

It’s

$$-\frac{1}{2}[1 -1 0]^T - \frac{9}{6}[1 1 -2]^T = [-2 -1 3]^T.$$
6. [10 points] Let $A$ be a matrix of order $n$ such that $3$ is an eigenvalue of $A$. Prove that $0$ is an eigenvalue of $A^2 - 4A + 3I_n$.

Let $x$ be an eigenvector of $A$ (so $x \neq 0$) for its eigenvalue $3$. Then

$$(A^2 - 4A + 3I_n)x = A^2x - 4Ax + 3x = A(3x) - 4(3x) + 3x = 9x - 12x + 3x = (9 - 12 + 3)x = 0x,$$

Since $x \neq 0$, $x$ is an eigenvector of the matrix $A^2 - 4A + 3I_n$ for the eigenvalue $0$. 

7. [17 points] Let $A$ be an $m$ by $n$ matrix and $b$ an $m$ by 1 vector such that $Ax = b$ has exactly one solution.

(i) **Prove** that the null space of $A$ equals $\{0\}$, that is, consists only of the zero vector.

Since $Ax = b$ has exactly one solution, the columns of $A$ must be linearly independent. Hence the nullspace, being the linear combinations of the columns which give the zero vector, of $A$ is $\{0\}$.

(ii) What is the rank of $A$?

Since the columns of $A$ are linearly independent, the rank must equal $n$.

(iii) **Prove** that $A^Ty = c$ must have a solution for every $n$ by 1 vector $c$. Since the rank of $A$ (and hence of the $n$ by $m$ matrix $A^T$), is $n$, the dimension of the column space of $A^T$ equals $n$. Thus every $n$-tuple $c$ is a linear combination of the columns of $A$. 
8. [20 points] We want to determine the quadratic curve \( y = a + bx + cx^2 \) that gives the best fit to the experimental data \((1, 2), (2, 5), (3, 6),\) and \((-1, 4)\).

(i) What system of linear equations \( Ax = b \) do we want the least squares solution of: specify \( A \) and \( b \).

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 4 \\
1 & 3 & 9 \\
1 & -1 & 1
\end{bmatrix}
\quad \text{and} \quad
b = \begin{bmatrix}
2 \\
5 \\
6 \\
4
\end{bmatrix}.
\]

(ii) Are the columns of \( A \) linearly independent? Why or why not?

Yes, because e.g. the first three rows give an invertible Vandermonde matrix.

(iii) What is the system of equations \( By = c \), a solution \( y \) of which gives a least squares solution of \( Ax = b \)? Compute \( B \) and \( c \) but you are not expected to solve \( By = c \).

\[
B = A^T A = \begin{bmatrix}
4 & 5 & 15 \\
5 & 15 & 35 \\
15 & 35 & 99
\end{bmatrix}
\quad \text{and} \quad
c = \begin{bmatrix}
17 \\
26 \\
80
\end{bmatrix}.
\]