Fall Semester, 2002-03
Math 743: Comments on solutions of Exercises 3; Due Monday, October 21, 2002.

1. Let $T$ be a linear transformation on the space of complex matrices of order $n$ such that $T$ preserves the spectrum (the eigenvalues, including multiplicities). Prove that there exists a nonsingular matrix $P$ such that $T(A) = P^{-1}AP$ for all $A$, or $T(A) = P^{-1}A^TP$ for all $A$.

**Proof** [This exercise was done well; in some cases there was one small error.] Since $T$ preserves eigenvalues it must preserve determinant and so is of the form $T(A) = PAQ$ (or $PA^TQ$) for some nonsingular matrices $P$ and $Q$. In particular, $T$ is nonsingular (a bijection on matrices of order $n$). With $C = (T(I))^{-1}$ ($T(I)$ must be nonsingular since $T$ preserves eigenvalues) we have

$$\det(\lambda I - A) = \det T(\lambda I - A) = \det(\lambda T(I) - T(A)) = \det T(I) \det(I - CA).$$

From this it follows that $\text{eig}(T(A)) = \text{eig}A = \text{eig}CA$ for all $A$. Since $T$ is nonsingular, every matrix of order $n$ is of the form $CX$ for some $X$, i.e. $\text{eig}(X) = \text{eig}(CX)$ for all $X$. By the polar decomposition there exist unitary $U$ and psdh $H$ such that $CU = H$. Now the eigenvalues of $U$ have absolute value 1, and those of $H$ are nonnegative reals. But by above $U$ and $H$ have the eigenvalues. So eigenvalues of $U$ and $H$ all equal 1. Since $U$ and $H$ are similar to diagonal matrices, they must be $I$ and $C = I$. So $T(I) = I$ and $PQ = PIQ = I$, that is, $Q = P^{-1}$.

2. Let $T$ be a linear transformation on the space of complex matrices of order $n$. Prove that $T$ preserves the singular values (including multiplicities) if and only if there are unitary matrices $P$ and $Q$ such that $T(A) = PAQ$ for all $A$, or $T(A) = PA^TQ$ for all $A$.

**Proof** Since $T$ preserves singular values and the rank is the number of nonzero singular values, $T$ preserves rank and so is of the form $T(A) = PAQ$ (or $PA^TQ$) for some nonsingular $P$ and $Q$.

Let $a$ and $b$ be singular values of $P$ and $Q$, respectively. Let $x$, $y$ be the unit vectors such that $Px = au$, and $y^*Q = bv^*$ where $x, y, u, v$ are unit vectors. Then $A = xy^*$ has rank 1 and so has singular values $1, 0, \ldots, 0$ (since
Ay = 1 \cdot x), and T(A) = Pxy^*Q is rank one with singular value \( ab, 0, \ldots, 0 \). So, \( ab = 1 \). From this we conclude that all singular values of \( P \) are the same and the same holds for \( Q \). Thus, after scaling, we can assume that \( P \) and \( Q \) are unitary.

3. Let \( A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \). Determine

a. The singular values, left singular (real) vectors, and right singular (real) vectors of \( A \).

b. Draw a careful picture of the unit ball in \( \mathbb{R}^2 \) and its image under \( A \), together with the singular vectors.

c. What are the 1-, 2-, \( \infty \)-, and Frobenius norms of \( A \)?

d. The inverse of \( A \) from the SVD.

[This was computational and caused no problem.]