This paper will introduce Cohomology of Sheaves, Cohomology of Affine Scheme, Čech Cohomology and Cohomology of Projective Space, then examine and compare their results.

Preliminary

In this paper some important definitions and basic properties of derived functors are assumed, with simply display the following vocabularies.

\( \mathbb{Ab} \): the category of abelian groups.
\( \text{Mod}(A) \): the category of modules over a commutative ring \( A \) with identity.
\( \mathbb{Ab}(X) \): the category of shaves of abelian groups on a topological space \( X \).
\( \text{Mod}(X) \): the category of sheaves of \( \mathcal{O}_X \)-modules on a ringed space \((X, \mathcal{O}_X)\).
\( \text{Qco}(X) \): the category of quasi-coherent sheaves of \( \mathcal{O}_X \)-modules on a scheme \( X \).
\( \text{Coh}(X) \): the category of coherent sheaves of \( \mathcal{O}_X \)-modules on a scheme \( X \).

Let \( A \) be an abelian category. An object \( I \) of \( A \) is injective if it satisfies the following universal property. Given an injection \( f : A \to B \) and a map \( \alpha : A \to I \), there exists at least one map \( \beta : B \to I \) such that \( \alpha = \beta \circ f \).

\[
\begin{array}{c}
\xymatrix{ 0 & A \ar[r]^-f & B \ar[d]_{\exists \beta} \ar@{.>}[lu]_{\alpha} \\
& I & }
\end{array}
\]

We say \( A \) has enough injectives if every object \( A \) in \( A \) there is an injection \( A \to I \) with \( I \) injective.

Now, let \( A \) be an abelian category with enough injectives, and let \( F : A \to B \) be a covariant left exact functor. Then we construct the right derived functor \( R^iF \), \( i \geq 0 \), of \( F \) as follows. For each \( A \) in \( A \), choose once and for all an injective resolution \( I \) of \( A \). Then we define \( R^iF(A) = h^i(F(I)) \).

**Theorem 0** Let \( A \) be an abelian category with enough injectives, and let \( F : A \to B \) be a covariant left functor to another abelian category \( B \). Then

(a) For each \( i \geq 0 \), \( R^iF \) as defined above is an additive functor from \( A \) to \( B \). Furthermore, it is independent (up to natural isomorphism of functors) of the choices of injective resolutions made.

(b) There is a natural isomorphism \( F \cong R^0F \).

(c) For each short exact sequence \( 0 \to A' \to A \to A'' \to 0 \) and for each \( i \geq 0 \) there is a natural morphism \( \delta^i : R^iF(A) \to R^{i+1}F(A') \), such that we obtain a long exact sequence

\[
\cdots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \to R^{i+1}F(A) \to \cdots
\]
§1. Cohomology of Sheaves

Proposition 1 If $A$ is a ring, every $A$-module is isomorphic to a submodule of an injective $A$-module. □

Proposition 2 Let $(X, O_X)$ be a ringed space. The category $\mathcal{Mod}(X)$ of sheaves of $O_X$-modules has enough injectives.

Proof

Let $\mathcal{F}$ be a sheaf of $O_X$-modules. We want to show that there is a injection $\mathcal{F} \rightarrow \mathcal{I}$ with $\mathcal{I}$ injective. For each point $x \in X$, the stalk $\mathcal{F}_x$ is an $O_{x,X}$-module. By Proposition 1, there is an injection $\mathcal{F}_x \rightarrow I_x$ where $I_x$ is an injective $O_{x,X}$-module. Let $\mathcal{M}_x$ be a skyscraper sheaf at $x \in X$ containing $I_x$. We have a morphism $\mathcal{F} \rightarrow \prod_{x \in X} \mathcal{M}_x$ which is injective.

We claim that $\mathcal{M}_x$ is injective. Given the injection $A \rightarrow B$ and $\alpha : A \rightarrow \mathcal{M}_x$.

\[
\begin{array}{ccc}
0 & \rightarrow & A \\
\alpha & \downarrow & \downarrow \\
& \mathcal{M}_x & \\
\end{array}
\]

We can extend $\alpha$ to $\beta : B \rightarrow \mathcal{M}_x$ as follows. For an open set $U$, if $x \notin U$, then $\Gamma(U, \mathcal{M}_x) = 0$ and the only extension is zero. If $x \in U$, then $\Gamma(U, \mathcal{M}_x) = \mathcal{M}_x$. Thus, $\mathcal{M}_x$ is injective. Finally, the product of injectives is injective, we are done.

Corollary 3 If $X$ is any topological space, then the category $\mathcal{Ab}(X)$ of sheaves of abelian groups on $X$ has enough injectives.

Proof

Let $O_X$ be the constant sheaf of rings $\mathbb{Z}$, then $(X, O_X)$ is a ringed space and $\mathcal{Mod}(X) = \mathcal{Ab}(X)$. □

Now, we are ready to define the cohomology functor. Let $X$ be a topological space. Let $\Gamma(X, \cdot)$ be the global section functor from $\mathcal{Ab}(X)$ to $\mathcal{Ab}$. We define the cohomology functor $H^i(X, \cdot)$ to be the right derived functors of $\Gamma(X, \cdot)$. So for any sheaf $\mathcal{F}$, the groups $H^i(X, \mathcal{F})$ are the cohomology groups of $\mathcal{F}$. i.e. Define

\[ R^i \Gamma(X, \mathcal{F}) = H^i(X, \mathcal{F}). \]

By the Theorem 0 in Preliminary, we have that $H^0(X, \mathcal{F}) = \Gamma(X)$ and $H^i(X, \mathcal{I}) = 0$ for $I$ injective and $i > 0$. And for every short exact sequence

\[ 0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0 \]

of sheaves, we have the long exact sequence

\[ 0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{G}(X) \rightarrow \mathcal{H}(X) \rightarrow H^1(X, \mathcal{F}) \rightarrow \cdots \]
**Definition** A sheaf \( F \) on a topological space \( X \) is *flasque* if for every inclusion of open sets \( V \subseteq U \), the restriction map \( F(U) \to F(V) \) is surjective.

**Lemma 4** If \((X, O_X)\) is a ringed space, any injective \( O_X \)-module is flasque.

**Proof**

Let \( U \subseteq X \) be an open set. Let \( O_U \) denote the sheaf which is the restriction of \( O_X \) to \( U \), extended by zero outside \( U \) (this means that \( O_U(V) = O_U \) for \( V \subseteq U \) and \( O_U = 0 \) if \( V \) is not in \( U \)). Now, let \( I \) be an injective \( O_X \)-module and \( V \subseteq U \) be an open set, we have an inclusion \( 0 \to O_V \to O_U \) of sheaves of \( O_X \)-modules. Since \( I \) is injective, the map \( O_V \to I \) can be extended to \( O_U \to I \), i.e. we have a surjection \( \text{Hom}(O_U, I) \to \text{Hom}(O_V, I) \to 0 \). But \( \text{Hom}(O_U, I) = I(U) \) and \( \text{Hom}(O_V, I) = I(V) \) (this comes from the fact that \( \text{Hom}_{\text{R}}(R, M) \cong M \)). Thus we the map \( I(U) \to I(V) \) is surjective. □

**Proposition 5** If \( F \) is a flasque sheaf on a topological space \( X \), then \( H^i(X, F) = 0 \) for all \( i > 0 \).

**Proof**

By Proposition 1, we let \( F \) embed in an injective object \( I \) of \( \text{Ab} \) and let \( G \) be the quotient.

\[
0 \longrightarrow F \longrightarrow I \longrightarrow G \longrightarrow 0
\]

By Lemma 4 \( I \) is flasque. Since \( F \) is flasque by assumption, \( G \) is flasque by the following diagram. Let \( V \subseteq U \), we have

\[
\begin{array}{cccccc}
0 & \longrightarrow & F(U) & \longrightarrow & I(U) & \longrightarrow & G(U) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F(V) & \longrightarrow & I(V) & \longrightarrow & G(V) & \longrightarrow & 0
\end{array}
\]

So the last vertical restriction map is surjective because the first two are. Now, since \( F \) is flasque, we have an exact sequence on global sections (proof omitted here, see Hartshorne II Ex1.16(b)).

\[
0 \longrightarrow \Gamma(X, F) \longrightarrow \Gamma(X, I) \longrightarrow \Gamma(X, G) \longrightarrow 0
\]

Since \( H^i(X, I) = 0 \) for all \( i > 0 \), from the long exact sequence we have \( H^i(X, F) = 0 \) and \( H^i(X, F) = H^{i-1}(X, G) \) for all \( i \geq 2 \). But \( G \) is also flasque, so by induction on \( i \) the result follows. □

**Remark** This result implies that flasque sheaves are acyclic for the functor \( \Gamma(X, \cdot) \). Hence we can calculate cohomology using flasque resolution.

**Proposition 6** Let \((X, O_X)\) be a ringed space. Then the derived functors of the functor \( \Gamma(X, \cdot) \) from \( \text{Mod}(X) \) to \( \text{Ab} \) coincide with the cohomology functors \( H^i(X, \cdot) \).

**Proof**

Consider \( F = \Gamma(X, \cdot) \) as a functor from \( \text{Mod}(X) \) to \( \text{Ab} \). We calculate its derived functors by taking injective resolutions in the category \( \text{Mod}(X) \). i.e. Let \( F \) be a sheaf on \( X \). Take an injective resolution \( G \) of \( F \), i.e. \( F \to G^* \). By Proposition 4, injective is flasque, then by
Proposition 5, flasque is acyclic. So this resolution gives the usual cohomology functors. □

A Vanishing Theorem of Grothendieck -

Theorem 7 (Grothendieck) Let $X$ be a noetherian topological space of dimension $n$. Then for all $i > n$ and all sheaves of abelian groups $F$ on $X$, we have $H^i(X, F) = 0$. □

§2. Cohomology of a Noetherian Affine Scheme

In this section we will prove that if $X = \text{Spec } A$ is a noetherian affine scheme, then $H^i(X, F) = 0$ for all $i > 0$ and all quasi-coherent sheaves $F$ of $\mathcal{O}_X$-modules. The main point is to show that if $I$ is an injective $A$-module, then the sheaf $\tilde{I}$ on Spec $A$ is flasque.

Proposition 8 (Krull Theorem) Let $A$ be a noetherian ring, let $M \subseteq N$ be finitely generated $A$-modules, and let $a$ be an ideal of $A$. Then the $a$-adic topology on $M$ is induced by the $a$-adic topology on $N$. In particular, for any $n > 0$, there exists an $n' \geq n$ such that $a^n M \supseteq M \cap a^{n'} N$. □

Definition For any ring $A$, and any deal $a \subseteq A$, and any $A$-module $M$, we have defined the submodule $\Gamma_a(M)$ to be $\{ m \in M \mid a^n m = 0 \text{ for some } n > 0 \}$.

Lemma 9 Let $A$ be a noetherian ring, let $a$ be an ideal of $A$ and $I$ be an injective $A$-module. Then the submodule $J = \Gamma_a(I)$ is also an injective $A$-module. □

Lemma 10 Let $I$ be an injective module over a noetherian ring $A$. Then for any $f \in A$, the natural map of $I$ to its localization $I_f$ is surjective. □

Proposition 11 Let $I$ be an injective module over a noetherian ring $A$. Then the sheaf $\tilde{I}$ on $X = \text{Spec } A$ is flasque. □

Theorem 12 Let $X = \text{Spec } A$ be the spectrum of a noetherian ring $A$. Then for all quasi-coherent sheaves $F$ on $X$, and for all $i > 0$, we have $H^i(X, F) = 0$.

Proof

Given $F$, let $M = \Gamma(X, F)$. Take an injective resolution $0 \to M \to I^\bullet$ of $M$ in the category of $A$-module. Then we obtain an exact sequence of sheaves $0 \to \tilde{M} \to \tilde{I}^\bullet$ on $X$. Now $F = \tilde{M}$ (Hartshorne II Corollary 5.5) and each $\tilde{I}^\bullet$ is flasque by Proposition 11. So we can use this resolution of $F$ to calculate cohomology by remark after Proposition 5.

Applying the functor $\Gamma$, we recover the exact sequence of $A$-modules $0 \to M \to I^\bullet$. Thus, $H^0(X, F) = M$ and $H^i(X, F) = 0$ for $i > 0$. □

Corollary 13 Let $X$ be a noetherian scheme, and let $F$ be a quasi-coherent sheaf on $X$. Then $F$ can be embedded in a flasque, quasi-coherent sheaf $G$. □
Theorem 14 (Serre) Let $X$ be a noetherian scheme. Then the following conditions are equivalent:

(i) $X$ is affine;
(ii) $H^i(X, \mathcal{F}) = 0$ for all $\mathcal{F}$ quasi-coherent and all $i > 0$;
(iii) $H^1(X, \mathcal{F}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$.

Proof

(i) $\Rightarrow$ (ii) is Theorem 12. (ii) $\Rightarrow$ (iii) is trivial. And (iii) $\Rightarrow$ (i) can be proved by the following criterion.

Criterion for Affineness (Exercise II 2.17(b)) A scheme $X$ is affine if and only if there is a finite set of elements $f_1, \cdots, f_r \in A = \Gamma(X, \mathcal{O}_X)$ such that the open subsets $X_f$, are affine, and $f_1, \cdots, f_r$ generate the unit ideal in $A$.

§3. Čech Cohomology

In this section we construct the Čech cohomology groups for a sheaf of abelian groups on a topological space $X$, with respect to a given open covering of $X$. We will prove that if $X$ is a noetherian separated scheme, the sheaf is quasi-coherent, and the covering is an open affine covering, then these Čech cohomology groups coincide with the cohomology groups defined in section 1.

Let $X$ be a topological space, and let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of $X$. Fix a well-ordering of the index set $I$. For any finite set of indices $i_0, \cdots, i_p \in I$ we denote the intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ by $U_{i_0, \cdots, i_p}$.

Definition Let $X$ be a topological space and let $\mathcal{U}$ be an open covering of $X$. For any sheaf of abelian groups $\mathcal{F}$ on $X$, we define the $p$th Čech cohomology group of $\mathcal{F}$, with respect to the covering $\mathcal{U}$, to be

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = h^p(C'(\mathcal{U}, \mathcal{F}))$$

where $C'(\mathcal{U}, \mathcal{F})$ is a complex of abelian groups.

Lemma 15 For any $X, \mathcal{U}, \mathcal{F}$ as above, we have $\check{H}^0(\mathcal{U}, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$. □

Proposition 16 Let $X$ be a topological space, let $\mathcal{U}$ be an open covering, and let $\mathcal{F}$ be a flasque sheaf of abelian groups on $X$. Then for all $p > 0$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$. □

Lemma 17 Let $X$ be a topological space, and $\mathcal{U}$ an open covering. Then for each $p \geq 0$ there is a natural map, functorial in $\mathcal{F}$,

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \rightarrow H^p(X, \mathcal{F})$$

□.

Theorem 18 Let $X$ be a noetherian separated scheme, let $\mathcal{U}$ be an open affine cover of $X$, and let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. Then for all $p \geq 0$, the natural maps of Lemma 17 give isomorphisms

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \cong H^p(X, \mathcal{F})$$. 


Proof

For \( p = 0 \), this is the case of Lemma 15. For the general case, we embed \( \mathcal{F} \) in a flasque, quasi-coherent sheaf \( \mathcal{G} \), and let \( \mathcal{H} \) be the quotient:

\[
0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0.
\]

For each \( i_0 < \cdots < i_p \), the open set \( U_{i_0, \ldots, i_p} \) is affine since it is an intersection of affine open subsets of a separated scheme. Since \( \mathcal{F} \) is quasi-coherent, we therefore have an exact sequence

\[
0 \to \mathcal{F}(U_{i_0, \ldots, i_p}) \to \mathcal{G}(U_{i_0, \ldots, i_p}) \to \mathcal{H}(U_{i_0, \ldots, i_p}) \to 0
\]

of abelian groups by Theorem 12. Taking product, we find that the corresponding sequence of Čech complexes

\[
0 \to C^0(\mathcal{U}, \mathcal{F}) \to C^0(\mathcal{U}, \mathcal{G}) \to C^0(\mathcal{U}, \mathcal{H}) \to 0
\]

is exact. Hence, we obtain a long exact sequence of Čech cohomology groups. Since \( \mathcal{F} \) is flasque, its Čech cohomology vanishes for \( p > 0 \) by Proposition 16. Hence we obtained an exact sequence

\[
0 \to \check{H}^0(\mathcal{U}, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \mathcal{G}) \to \check{H}^0(\mathcal{U}, \mathcal{H}) \to \check{H}^1(\mathcal{U}, \mathcal{F}) \to 0
\]

and isomorphisms

\[
\check{H}^p(\mathcal{U}, \mathcal{H}) \cong \check{H}^{p+1}(\mathcal{U}, \mathcal{F})
\]

for \( p \geq 1 \).

Now, using the case when \( p = 0 \) and Proposition 5 to compare with the long exact sequence of usual cohomology for the above short exact sequence, we see that

\[
\check{H}^1(\mathcal{U}, \mathcal{F}) \to \check{H}^1(X, \mathcal{F})
\]

is an isomorphism. Since \( \mathcal{H} \) is also quasi-coherent, the result follows by induction on \( p \). \( \square \)

§4. The Cohomology of Projective Space

In this section we will explicitly calculate the cohomology of the sheaves \( \mathcal{O}(n) \) on a projective space, by using Čech cohomology for a suitable open affine covering. For the following content, we let \( A \) be a noetherian ring, let \( S = A[x_0, \ldots, x_r] \), and let \( X = \text{Proj} S \) be the projective space \( \mathbb{P}^r_A \) over \( A \). Let \( \mathcal{O}_X(1) \) be the twisting sheaf of Serre. For any sheaf of \( \mathcal{O}_X \)-modules \( \mathcal{F} \), we denote by \( \Gamma_*(\mathcal{F}) \) the graded \( S \)-module \( \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)) \).

Lemma 19 Let \( A \) be a noetherian ring, and let \( X = \mathbb{P}^r_A \) with \( r \geq 1 \). Then the natural map \( S \to \Gamma_*(\mathcal{O}_X) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n)) \) is an isomorphism. \( \square \).

Theorem 20 Let \( A \) be a noetherian ring, and let \( X = \mathbb{P}^r_A \) with \( r \geq 1 \). Then \( H^i(X, \mathcal{O}_X(n)) = 0 \) for \( 0 < i < r \) and all \( n \in \mathbb{Z} \).

Proof

Consider the exact sequence of graded \( S \)-modules

\[
0 \to S(-1) \xrightarrow{x_r} S \to S/(x_r) \to 0.
\]
This gives the exact sequence of sheaves

\[ 0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_H \to 0 \]

on X where H is the hyperplane \( x_r = 0 \). Twisting by all \( n \in \mathbb{Z} \) and taking the direct sum, we get

\[ 0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0 \]

where \( \mathcal{F}_H = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_H(n) \). Now, taking cohomology we get a long exact sequence

\[ \cdots \to H^i(X, \mathcal{F}(-1)) \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F}_H) \to \cdots \]

For \( i = 0 \), we have an exact sequence

\[ 0 \to H^0(X, \mathcal{F}(-1)) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}_H) \to 0 \]

by Lemma 19 since \( H^0(X, \mathcal{F}_H) \) is just \( S/(x_r) \).

Now, considered as graded \( S \)-modules, \( H^i(X, \mathcal{F}(-1)) \) is just \( H^i(X, \mathcal{F}) \) shifted one place, and the map \( H^i(X, \mathcal{F}(-1)) \to H^i(X, \mathcal{F}) \) of the exact sequence is multiplication by \( x_r \).

Now \( H \) is isomorphic to \( P_A^{r-1} \), and \( H^i(X, \mathcal{F}_H) = H^i(H, \bigoplus \mathcal{O}_H(n)) \). We see that \( H^i(X, \mathcal{F}_H) = 0 \) for \( 0 < i < r - 1 \) by induction hypothesis to \( \mathcal{F}_H \).

At the end of the exact sequence we have

\[ 0 \to H^{r-1}(X, \mathcal{F}_H) \xrightarrow{\delta} H^r(X, \mathcal{F}(-1)) \xrightarrow{x_r} H^r(X, \mathcal{F}) \to 0 \]

It is clear that \( x_r \) is surjective. On the other hand, the kernel of \( x_r \) is the free submodule generated by the negative monomials \( x_0^{l_0} \cdots x_r^{l_r} \) with \( l_r = 1 \). Note that \( H^{r-1}(X, \mathcal{F}_H) \) is the free \( A \)-module with basis consisting of negative monomials in \( x_0, \ldots, x_{r-1} \), and \( \delta \) is division by \( x \), the sequence is exact. Hence \( \delta \) is injective.

Therefore, the long exact sequence of cohomology shows that the map \( x_r : H^i(X, \mathcal{F}(-1)) \to H^i(X, \mathcal{F}) \) is bijective for \( 0 < i < r \). □

**Theorem 21** Let \( X \) be a projective scheme over a noetherian ring \( A \), and let \( \mathcal{O}_X(1) \) be a very ample invertible sheaf on \( X \) over Spec \( A \). Let \( \mathcal{F} \) be a coherent sheaf on \( X \). Then there is an integer \( n_0 \), depending on \( \mathcal{F} \), such that for each \( i > 0 \) and each \( n \geq n_0 \), \( H^i(X, \mathcal{F}(n)) = 0 \).

Proof

Given a coherent sheaf \( \mathcal{F} \) on \( X \), we can write \( \mathcal{F} \) as a quotient of a sheaf \( \mathcal{E} \), which is a finite direct sum of sheaves \( \mathcal{O}(q_i) \) for various integer \( q_i \). Let \( \mathcal{R} \) be the kernel,

\[ 0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0. \]

Then \( \mathcal{R} \) is also coherent. We get an exact sequence of \( A \)-modules

\[ \cdots \to H^i(X, \mathcal{E}) \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{R}) \to \cdots \]

Now, we twist the sequence by \( \mathcal{O}(n) \) we get

\[ \cdots \to H^i(X, \mathcal{E}(n)) \to H^i(X, \mathcal{F}(n)) \to H^{i+1}(X, \mathcal{R}(n)) \to \cdots \]

Now for \( n \gg 0 \), the module on the left vanishes because \( \mathcal{E} \) is a sum of \( \mathcal{O}(q_i) \). The module on the right also vanishes for \( n \gg 0 \) by the induction hypothesis. Hence, \( H^i(X, \mathcal{F}(n)) = 0 \) for \( n \gg 0 \). Finally, since there are only finitely many \( i \) involved in the statement of the Theorem,
i.e. $0 < i \leq r$, it is sufficient to determine $n_0$ separately for each $i$. We are done. □

**Definition** An invertible sheaf $\mathcal{L}$ on a noetherian scheme $X$ is said to be *ample* if for every coherent sheaf $\mathcal{F}$ on $X$, there is an integer $n_0 > 0$ (depending on $\mathcal{F}$) such that for every $n \gg n_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^n$ is generated by its global sections.

**Remark:** Note the ”ample” is an absolute notion, i.e., it depends only on the scheme $X$, whereas ”very ample” is a relative notion, depending on a morphism $X \to Y$.

**Lemma 22** Let $X$ be a scheme of finite type over a noetherian ring $A$, and let $\mathcal{F}$ be an invertible sheaf on $X$. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is very ample over Spec $A$ for some $m > 0$. □

**Theorem 23** Let $A$ be a noetherian ring, and let $X$ be a proper scheme over Spec $A$. Let $\mathcal{L}$ be an invertible sheaf on $X$. If in addition $\mathcal{L}$ is ample, then for each coherent sheaf $\mathcal{F}$ on $X$, there is an integer $n_0$, depending on $\mathcal{F}$, such that for each $i > 0$ and each $n \gg n_0$, $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$.

Proof

If $\mathcal{L}$ is ample on $X$, then for some $m > 0$, $\mathcal{L}^m$ is very ample on $X$ over Spec $A$ by Lemma 22. Since $X$ is proper over Spec $A$, it is necessarily projective. Then applying Theorem 21 to each of the sheaves $\mathcal{F}, \mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \mathcal{L}^2, \cdots, \mathcal{F} \otimes \mathcal{L}^{m-1}$ the result follows. □

§5. Summary

We have used different ways of introducing cohomology, yet the result is the same. We took out basic definition the derived functors of the global section functor in §1. This definition is more general and most suitable for theoretical questions. But it is impossible to calculate in practice. Thus Čech cohomology is introduced in §3. And in §4 we use it to compute cohomology of sheaves $\mathcal{O}(n)$ on projective space $\mathbb{P}^r$ explicitly.

In §2, under the case of noetherian we proved that the higher cohomology of quasi-coherent sheaf on an affine scheme is zero. Then we used this to prove that Čech cohomology agree with the derived functor cohomology. This is technically simpler than the case of arbitrary affine scheme. Therefore, we were proving all theorems under the constraint of the noetherian hypothesis.

References


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