Worksheet 7 Exercise solutions

1. \[ \int \frac{2}{x^2 - x} \, dx \]

**Intuition:** For \( x > 2 \), \( \frac{1}{x} \) is greater than 0, so no problem points except at \( \pm \infty \). Since this behaves similarly to \( \frac{1}{x^2} \), we should expect this to converge.

\[ \frac{2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \]

\[ a=2 \]
\[ b=2 \]

\[ \lim_{b \to \infty} \int_{0}^{b} \frac{2}{x^2 - x} \, dx = \lim_{b \to \infty} \int_{0}^{b} \left( \frac{1}{x} - \frac{1}{x-1} \right) \, dx = \lim_{b \to \infty} \left[ \ln |x-1| - \ln |x| \right]_{a}^{b} \]

\[ = \lim_{b \to \infty} \left[ 2 \ln \left( \frac{b-1}{b} \right) - 2 \ln \left( \frac{2-1}{2} \right) \right] \]

\[ = \lim_{b \to \infty} \left[ 2 \ln \left( \frac{b}{b-1} \right) - 2 \ln \left( \frac{1}{2} \right) \right] \]

\[ = 0 + \ln \left( \frac{1}{2} \right)^2 = \ln 4 \]

2. \[ \int_{0}^{\infty} \frac{d\theta}{1 + e^{\theta}} \]

**Intuition:** This has only a problem point at \( \pm \infty \). It behaves like \( \frac{1}{e^\theta} \), so we should expect it to converge.

We apply comparison theorem: Note that \( 1 + e^\theta > e^\theta \). Thus \( \frac{1}{1 + e^\theta} < \frac{1}{e^\theta} \).

Since \( \int_{0}^{\infty} \frac{d\theta}{e^\theta} \) converges (easy calculation), \( \int_{0}^{\infty} \frac{d\theta}{1 + e^\theta} \) also converges by the comparison theorem.

3. \[ \int_{0}^{1} \ln x \, dx \]

**Intuition:** Problem pt. at 0 since \( \lim_{x \to 0} \ln x = -\infty \). We can integrate by parts by IBP, so we can compute this directly by taking a limit.

\[ = \lim_{a \to 0} \int_{a}^{1} \ln x \, dx = \lim_{a \to 0} \left[ \frac{x}{2} \ln x \right]_{a}^{1} - \int_{a}^{1} \frac{1}{2} \, dx = \lim_{a \to 0} \left[ \frac{x}{2} \ln x - \frac{x}{2} \right]_{a}^{1} \]

\[ = \lim_{a \to 0} \left[ \left( 0 - \frac{1}{2} \right) - \left( \frac{1}{2} \ln a - \frac{a}{2} \right) \right] \]

\[ = -\frac{1}{4} \]

\[ \text{as } a \to 0 \text{ by l'Hôpital.} \]
\[ \int_{e}^{\infty} \ln(\ln x) \, dx \]

Intuition: as \( x \to \infty \), \( \ln x \to \infty \) and so \( \ln(\ln x) \to \infty \) as well. Thus the integrand keeps growing, so we cannot expect this to converge.

Use comparison test to show divergence.

Note that if \( x \geq e^a \), \( \ln(\ln x) \geq 1 \). Furthermore, \( \int_{2}^{\infty} 1 \, dx \) diverges.

Thus \( \int_{e}^{\infty} \ln(\ln x) \, dx \) diverges by the comparison theorem.

\[ \int_{0}^{\infty} \frac{dx}{\sqrt{x^6+1}} \]

Intuition: This has a problem point only at \( \infty \). It behaves like \( \frac{1}{\sqrt{x^6}} = \frac{1}{x^3} \), so we expect it to converge. However, \( \int_{0}^{\infty} \frac{1}{x^3} \, dx \) does not converge so we'll have to use the tail theorem to get away from \( 0 \) before making our comparison.

Note that for \( 0 < x < \infty \), \( \frac{1}{\sqrt{x^6+1}} \) is continuous. Thus the tail theorem says that

\[ \int_{0}^{\infty} \frac{dx}{\sqrt{x^6+1}} \text{ converges if and only if } \int_{1}^{\infty} \frac{dx}{\sqrt{x^6+1}} \text{ converges.} \]

We use the comparison theorem to show convergence of \( \int_{1}^{\infty} \frac{dx}{\sqrt{x^6+1}} \). Note that for \( x \geq 1 \), \( \sqrt{x^6+1} \geq \sqrt{x^6} = x^3 \).

Thus \( \frac{1}{\sqrt{x^6+1}} \leq \frac{1}{x^3} \). Since \( \int_{1}^{\infty} \frac{1}{x^3} \, dx \) converges, so does \( \int_{1}^{\infty} \frac{1}{\sqrt{x^6+1}} \, dx \) by the comparison theorem.

Thus, by the tail theorem \( \int_{0}^{\infty} \frac{dx}{\sqrt{x^6+1}} \) also converges.
6. \[ \int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, dx \] 

Intuition: The problem point is at \( x = 0 \), since the numerator \( e^{-\sqrt{x}} \) is bounded between 1 and \( \frac{1}{e} \), this behaves like \( \frac{1}{\sqrt{x}} \), which we know converges. Thus we should expect convergence.

If we just want to show convergence, we can use the comparison theorem.

Note that if \( 0 \leq x \leq 1 \), \( e^{-\sqrt{x}} \leq 1 \) and \( e^{\sqrt{x}} \geq 0 \). Thus \[ \frac{e^{-\sqrt{x}}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}}. \]

We know \( \int_0^1 \frac{1}{x^{1/2}} \, dx \) converges, thus \( \int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, dx \) also converges by comparison theorem.

[Note that we can also compute this directly by integrating \( e^{-\sqrt{x}} \) using a substitution.]

7. \[ \int_0^\infty \frac{dx}{e^x + e^{-x}} \]

Intuition: First note that \( f(x) = \frac{1}{e^x + e^{-x}} \) is an even function, that is \( f(x) = f(-x) \). Thus, \[ \int_0^\infty \frac{dx}{e^x + e^{-x}} = 2 \int_0^\infty \frac{dx}{e^x + e^{-x}}. \] This behaves like \( 1/e^x \), so we should expect this to converge.

If we just want to show convergence, we can just use comparison theorem.

For \( x \geq 0 \).

Since \( e^x > 0 \), \( e^{-x} + e^x \geq e^x \). Thus \[ \frac{1}{e^x + e^{-x}} \leq \frac{1}{e^x}. \] We know \( \int_0^\infty \frac{1}{e^x} \, dx \) converges, so \( \int_0^\infty \frac{1}{e^x + e^{-x}} \, dx \) also converges. Since \[ \int_0^\infty \frac{1}{e^x + e^{-x}} \, dx = 2 \int_0^\infty \frac{dx}{e^x + e^{-x}} \], this also converges.

8. \[ \int_2^\infty \frac{dx}{\ln x} \] 

Intuition: \( \frac{1}{\ln x} \) decays more slowly than \( \frac{1}{x} \), so we should expect this to diverge.

Use comparison theorem: If \( x \geq 2 \), then \( \ln x \leq x \). Thus \[ \frac{1}{\ln x} \geq \frac{1}{x}. \]

Since \( \int_2^\infty \frac{1}{x} \, dx \) diverges, so does \( \int_2^\infty \frac{1}{\ln x} \, dx \).