Math 234 - Worksheet 1 Solutions

1. Find the slope of the tangent line to the graph of the function \( f(x) = \frac{1}{x^2 + 4x + 4} \) at \( x = 0 \).

**Solution:** The slope of the tangent line at \( x = 0 \) is given by the value of the derivative \( f' \) at \( x = 0 \). We can rewrite \( f(x) = (x + 2)^{-2} \). We can then apply the power rule to get that

\[
f'(x) = -2(x + 2)^{-3} = \frac{-2}{(x + 2)^3}.
\]

Evaluating this at \( x = 0 \), we have that the slope is \( \frac{-1}{4} \).

2. Find the following limit:

\[
\lim_{x \to -4} \frac{x^2 + 7x + 12}{x^2 - 16}.
\]

**Solution:** Since we can’t simply plug in the value \( x = -4 \), we can first factor both the numerator and denominator and cancel out common terms to get

\[
\lim_{x \to -4} \frac{x^2 + 7x + 12}{x^2 - 16} = \lim_{x \to -4} \frac{(x + 3)(x + 4)}{(x - 4)(x + 4)} = \lim_{x \to -4} \frac{x + 3}{x - 4} = \frac{1}{8}.
\]

3. Find the following limit or show that it does not exist:

\[
\lim_{x \to 0} \frac{\sin(3x^2)}{4x^2}.
\]

**Solution:** We use the substitution \( u = 3x^2 \) and note that as \( x \to 0 \) we also have that \( u \to 0 \). We can also write \( 4x^2 = \frac{4}{3}u \). Thus,

\[
\lim_{x \to 0} \frac{\sin(3x^2)}{4x^2} = \lim_{u \to 0} \frac{\sin(u)}{4u/3} = \frac{3}{4} \lim_{u \to 0} \frac{\sin(u)}{u}.
\]

Using the fact that \( \lim_{u \to 0} \frac{\sin(u)}{u} = 1 \), we then have that our limit is \( \frac{3}{4} \).
4. Find the global maximum and minimum values of the function

\[ f(x) = \frac{x^2 - 2x + 1}{x^2 + 1} \]

defined on the the interval \([-5, 5]\) and also indicate all the points of local extrema.

**Solution:** We calculate the derivative of \(f\) using the quotient rule to get

\[ f'(x) = \frac{2x^2 - 2}{(x^2 + 1)^2}. \]

Since local maxima and minima occur at either the endpoints of the interval or where the derivative is zero, we want to solve \(f'(x) = 0\). Since the denominator is always positive, the derivative is zero if and only if the numerator is zero. We simply need to find the solutions to \(2x^2 - 2 = 0\), which are \(x = 1\) and \(x = -1\). Furthermore, we can easily check that the derivative is positive on the intervals \([-5, -1)\) and \((1, 5]\) and negative on the interval \((-1, 1)\).

Thus there are local minima at \(x = -5\) and \(x = 1\) and local maxima at \(x = -1\) and \(x = 5\). We can plug these values into \(f\) to determine which of these is the global maximum and which is the global minimum.

Since \(f(-1) = 2\) and \(f(5) = \frac{8}{13}\), the global maximum occurs at \(x = -1\). Since \(f(-5) = \frac{18}{13}\) and \(f(1) = 0\), the global minimum occurs at \(x = 1\).

Ask wolframalpha to plot this function to see what it looks like.

5. If a rectangle is inscribed in a semicircle of radius 2, what is the largest area this rectangle can have? What are the dimensions of this rectangle?

**Solution:** I would suggest first drawing a diagram of this problem. If the rectangle intersects the semicircle at the points \((-x, y)\) and \((x, y)\), we can then see that the area of this rectangle is given by \(A = 2xy\). We want to make this a function of a single variable, so we can use the fact that \((x, y)\) is located on the semicircle of radius two to solve for \(y\) in terms of \(x\). By the Pythagorean theorem, we have

\[ x^2 + y^2 = 2^2, \]

which then implies \(y = \sqrt{4 - x^2}\). This means we can write the formula for the area as a function of the single variable \(x\). We have

\[ A(x) = 2x\sqrt{4 - x^2}. \]

We can then take the derivative using the product rule (don’t forget to use the chain rule!) to get

\[ A'(x) = 2\sqrt{4 - x^2} + 2x \cdot \frac{-2x}{2\sqrt{4 - x^2}} = \frac{8 - 4x^2}{\sqrt{4 - x^2}}. \]

Since the maximum will occur when the derivative is zero, we set \(A'(x) = 0\) and solve to get \(x = \sqrt{2}\) and \(x = -\sqrt{2}\). Since we want \(x\) to be positive, we have that the area is maximized when \(x = \sqrt{2}\). Thus the dimensions of the rectangle with maximum area are a base of \(2\sqrt{2}\) and height of \(\sqrt{2}\). This rectangle has area of 8.
6. Compute the area of the region bounded by \( y = 2x^2 + 10 \) and \( y = 4x + 16 \).

**Solution:** Begin by making a rough sketch of the graphs of the functions. We want to determine where these two functions intersect. We set the two equal and solve, which gives us

\[
2x^2 + 10 = 4x + 16 \quad \Rightarrow \quad 2(x - 3)(x + 1) = 0 \quad \Rightarrow \quad x = -1, x = 3.
\]

The region bounded by the graphs is above the interval \([-1, 3]\). Furthermore, on this interval the function \( y = 4x + 16 \) is greater than the function \( y = 2x^2 + 10 \). Thus the area is given by the integral

\[
A = \int_{-1}^{3} \left( (4x + 16) - (2x^2 + 10) \right) dx = \frac{64}{3}.
\]

7. Compute the volume of the solid of revolution generated by rotating the region bounded by the graphs of \( y = \frac{x}{2} \) and \( y = \sqrt{x} \) about the \( x \)-axis.

**Solution:** Begin by making a rough sketch of the graphs of the two functions. Setting the two equations equal and solving, we can determine their intersection points are at \( x = 0 \) and \( x = 4 \). The volume is then given by the integral

\[
V = \pi \int_{0}^{4} \left( (\sqrt{x})^2 - (\frac{x}{2})^2 \right) dx = \frac{8\pi}{3},
\]

where the outer radius is the function \( y = \sqrt{x} \) and the inner radius is \( y = \frac{x}{2} \).

8. Compute the following integrals:

(a) \( \int \frac{\sin x}{4 + \cos^2 x} dx \),

(b) \( \int \frac{\tan x}{4 \sec x + \cos x} dx \).
9. Compute the following integrals:

(a) \( \int x^2 \ln(x/2) \, dx \),

(b) \( \int e^{2x} \sin x \, dx \).

10. Compute the Taylor series expansion of the function \( f(x) = x^4 e^{-3x^2} \) at the point \( x = 0 \).
11. Given the points \( A = (-1, -1), B = (2, 0), C = (3, -1), \) and \( D = (0, -2), \) is \( ABCD \) a parallelogram? Provide justification for your answer.

12. Find a parametric equation for the line \( \ell \) passing through the points \( A = (2, 3, 4) \) and \( B = (2, 1, 2) \). Determine which coordinate planes \( \ell \) intersects and find the intersection points.