Categories of Equivariants

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Introduction

In this article, $G$ will be a finite group and $k$ will be a field. In section 1, I will define what it means for $G$ to have an action on a $k$-linear category, and I will define the category of equivariant objects along with some of its basic properties. In section 2, I will assume that the group $G$ is abelian, and that the natural map $G \to \text{Hom}(\text{Hom}(G, k^*), k^*)$ is an isomorphism. This will allow for an important duality result. Section 3 is concerned with the interaction between $G$ actions and derived categories. In section 4, I will give a sample application in the case of an isogeny of abelian varieties. Many thanks to Dima Arinkin for suggesting this topic and for giving much helpful advice.

1 Equivariant Categories

Recall that a $k$-linear category is one enriched over $k$-vector spaces and containing all finite biproducts and a 0 object.

Definition 1. We say that $G$ acts on $C$ on the right if we have:

i. For every $g \in G$ a $k$-linear functor $F_g : C \to C$, that $F_1 = id_C$. If the category $C$ is abelian or has a triangulated structure, it should be assumed that the action functors respect this structure unless state otherwise.
ii. For every \( g, h \in G \) a natural isomorphism \( \Delta_{g,h} : F_h \circ F_g \to F_{gh} \) which together satisfy the coherence condition

\[
(\Delta_{gh,k}) \circ (F_k(\Delta_{g,h})) = (\Delta_{g,hk}) \circ (\Delta_{h,k}|F_g)
\] (1)

A better way to state this coherence is to consider the quadrilateral:

![Diagram of quadrilateral]

There are two ways to decompose the quadrilateral into pairs of triangles and these correspond to the two ways to get natural isomorphisms \( F_k \circ F_h \circ F_g \to F_{ghk} \); coherence says these should be equal.

**Example 2.** One can check that this definition is consistent with the situation when \( C \) is the category of sheaves of \( k \) vector spaces on a topological space \( X \) which carries a continuous, left \( G \) action. Then the pullback functors \( g^* \) for \( g \in G \) act on \( \text{Sh}_X \) on the right.

Given a category with \( G \) action, we can look for objects which are isomorphic to their images under the action functors, but the proper notion is that of an equivariant object which involves an extra structure.

**Definition 3.** We say that \( A \) is an **equivariant object** for the action of \( G \) on \( C \) if, for all \( g \) in \( G \) we have isomorphisms \( \phi_g : A \to F_g(A) \) which satisfy the coherence:

\[
\Delta_{g,h}(A) \circ F_h(\phi_g) \circ \phi_h = \phi_{gh}
\] (2)

**Example 4.** These isomorphisms are important. Consider the case when the action of \( G \) is trivial and when the \( \Delta \) isomorphisms are trivial. Then for an object \( A \) to have a \( G \) equivariant structure is exactly to have a representation \( G \to \text{Aut}(G,G) \).

Given two objects with equivariant structure \( (A, \phi_\bullet), (B, \psi_\bullet) \), we define a morphism of equivariant objects to be a \( C \)-morphism \( f : A \to B \) such that

\[
\psi_g \circ f = F_g(f) \circ \phi_g
\] (3)
for all \( g \in G \). To state this another way, the vector space \( \text{Hom}_C(A, B) \) carries a right \( G \) action which sends a morphism \( f \) to \( \psi^{-1}_g \circ F_g(f) \circ \phi_g \). Then the morphisms as equivariant objects are the \( G \) invariant \( C \) morphisms with respect to this action:

\[
\text{Hom}_{C^G}((A, \phi_*), (B, \psi_*)) = \text{Hom}_C(A, B)^G
\]  

(4)

**Definition 5.** The category \( C^G \), called the category of equivariant objects (of \( C \)) or the equivariant category for short, is the category whose objects are objects of \( C \) with \( G \) equivariant structures and whose morphisms are morphisms of equivariant objects.

Since group actions on categories involve more structures than, say, group actions on vector spaces or sets, it will be convenient to have the following technical lemmas stated here without proof:

**Lemma 6.** Suppose that \( F^\bullet \) is another set of functors parametrized by \( G \) and that for all \( g \) in \( G \), \( \eta_g : F^g \to F_g \) is a fixed natural isomorphism. Then the collection \( F^\bullet \) form another action of \( G \) on \( C \). Let \( C^G \) be the equivariant category for this new action, then the isomorphisms \( \eta_* \) can be used to construct an equivalence \( C^G \cong C'^G \).

**Lemma 7.** Suppose that \( C \xrightarrow{q,q^{-1}} E \) is an equivalence of categories. Then the \( G \) action on \( C \) may be pulled back to \( E \). The functors \( q, q^{-1} \) yield an equivalence of categories \( C^G \cong E^G \).

These facts will be handy later, but their proofs are unenlightening.

There are two important functors between the categories \( C \) and \( C^G \). The first is the forgetful functor \( \Phi : C^G \to C \). If \( A \) is an object of \( C^G \), I will sometimes write \( |A| \) for \( \Phi(A) \). The second functor \( \Sigma : C \to C^G \), called the averaging or induction functor, sends \( A \) to \( \bigoplus_G F_g(A) \) and a natural equivariant structure is given by the transformations \( \Delta_{g,*} \). For

\[
\Sigma(A) = \bigoplus_{x \in G} F_{xg}(A) \quad F_g(A) = \bigoplus_{x \in G} F_g(F_x(A))
\]

and the isomorphism \( \Sigma(A) \to F_g(\Sigma(A)) \) is given by

\[
\bigoplus_{x \in G} [\Delta_{xg}(A)^{-1} : F_{xg}(A) \to F_g(F_x(A))]
\]

**Proposition 8.** The category \( C^G \) inherits a \( k \)-linear structure in an obvious way. In particular, \( \Phi \) and \( \Sigma \) are \( k \)-linear functors. If \( C \) is abelian, then \( C^G \) is too, and the functors \( \Phi \) and \( \Sigma \) are exact.
Proof. By (4), \( C^G \) is enriched over \( k \)-vector spaces in such a way that \( \Phi \) is \( k \)-linear. Since it is constructed from the functors \( F_* \), the functor \( \Sigma \) also respects the \( k \)-linear structure. Given two objects of \( C^G \), their biproduct in \( C \) gets the obvious equivariant structure and satisfies the universal property of biproduct in \( C^G \). Both \( \Phi \) and \( \Sigma \) respect this biproduct structure.

Suppose now that \( C \) is an abelian category. Let

\[
A \overset{f}{\rightarrow} B \rightarrow C
\]

be a cokernel diagram, with \( C \) the cokernel of \( f \). By the universal property of cokernel applied to this diagram, \( C \) has a unique map to \( F_g(C) \) commuting in the diagram:

\[
\begin{array}{ccc}
F_g(A) & \xrightarrow{F_g(f)} & F_g(B) \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B \\
\end{array} \quad \begin{array}{ccc}
& & \\
& & \\
\downarrow & & \downarrow \\
& & \\
& & \\
F_g(C) & & C
\end{array}
\]

Thus \( C \) has maps all of its images under the functors \( F_g \). By the uniqueness part of the cokernel universal property, these maps must satisfy the coherence (2). Hence the \( C \) cokernel of \( f \) gets an equivariant structure which is compatible with that on \( B \) by construction. This shows that \( C^G \) has cokernels, and kernels follow similarly. This shows that \( C^G \) is preabelian.

If \( f \) is a morphism of \( C \), the cokernel of its kernel has a map \( \overline{f} \) induced by universal property to the kernel of the cokernel of \( f \), and this map is an isomorphism if \( C \) is abelian. The same map \( \overline{f} \) is induced by universal property if this diagram is in \( C^G \). After reducing by \( \Phi \), the maps \( \overline{f} \) coincide, as they were induced by the same universal properties. So \( \overline{f} \) is an isomorphism in both categories, which shows that \( C^G \) is abelian. \( \square \)

Example 9. As an example of this inheritance of structure, consider the case of a \( G \)-torsor \( X \overset{\pi}{\rightarrow} Y \), when \( X \) and \( Y \) are \( k \)-varieties. This morphism is flat and so flat descent applies from \( \text{Coh}(X) \) to \( \text{Coh}(Y) \). However, since \( X \) is a \( G \)-torsor on \( Y \), a descent datum on an object of \( \text{Coh}(X) \) is the same as a \( G \)-equivariant structure, and in fact the descent category is equivalent to the category of equivariants. But since descent is effective, the category of equivariants of \( \text{Coh}(X) \) is equivalent to the category of modules \( \text{Coh}(Y) \). This illustrates the inheritance of abelian structure.
There is a further structure which the category $C^G$ inherits that is unmentioned so far. Let $\chi$ be a character of $G$ and let $(A, \phi)$ be an equivariant object. Then we can twist $\phi$ by $\chi$. Namely, define $\phi'_g = \phi_g \chi(g)$. Then I claim that $\phi'_* = \phi_g \chi(g)$ is a new equivariant structure for $A$. I just need to verify that (2) is still satisfied:

$$
\Delta_{g,h}(A) \circ F_h(\phi'_g) \circ \phi'_h = \Delta_{g,h}(A) \circ F_h(\chi(g) \phi_g) \circ \chi(h) \phi_h \\
= \Delta_{g,h}(A) \circ F_h(\phi_g) \circ \phi_h \chi(g) \chi(h) \\
= \phi_{gh} \chi(g h) = \phi'_{gh}
$$

Even better is the following:

**Lemma 10.** The ‘twist’ by $\chi$ extends to an auto equivalence $F_\chi$ of $C^G$. Further, the collection of these twists form a group action of $\hat{G} = \text{Hom}(G, k^*)$ on $C^G$.

**Proof.** I want to show that this twist by $\chi$ extends to a functor $F_\chi : C^G \to C^G$. So let $f : (A, \phi) \to (B, \psi)$ be a morphism of equivariant objects. Then for all $g \in G$, I have $\psi_g \circ f = F_g(f) \circ \phi_g$. After multiplying both sides by $\chi(g)$, it is clear that $\psi'_g \circ f = F_g(f) \circ \phi'_g$.

This gives a candidate definition for $F_\chi$. Note that the functor $F_\chi$ does nothing to morphisms of $C^G$ – if $f : A \to B$ commutes with equivariant structures on $A$ and $B$, then $f$ still commutes with the $\chi$ twisted structures. Because this operation sends identity functions to identity functions and respects composition, $F_\chi$ is an endofunctor of $C^G$.

Now I need to check that the collection of $F_\chi$ as $\chi$ ranges over $\hat{G}$ form an action. But in this case it is unusually easy to see that the $F_\chi$ have appropriate natural transformations which satisfy (1) because they are actually exactly multiplicative. That is $F_{\chi_2} \circ F_{\chi_1} = F_{\chi_1 \chi_2}$. This shows that $\hat{G}$ has a well defined action on $C^G$. 

This construction is central to the next section. There is one more basic fact I want to have before moving on to the next section - it concerns the two functors $\Phi$ and $\Sigma$ we have mentioned:

**Lemma 11.** There exist adjunctions $\Phi \dashv \Sigma$ and $\Sigma \dashv \Phi$. Together these adjunctions give natural transformations $\text{id}_C \to \Phi \circ \Sigma$ and $\Phi \circ \Sigma \to \text{id}_C$. The composition of these two natural transformations is the natural endomorphism of $\text{id}_C$ given by $|G|$ times the identity natural automorphism of $\text{id}_C$.

**Proof.** Let $M \in C$ and let $(N, \phi_*) \in C^G$. Let $f : M \to N$ be a morphism of $C$. Now $M$ is a summand of $\Sigma(M)$, and I will choose to map that summand to $N$ by $f$. Now, in order to
satisfy (3), it is necessary to map the $F_g(M)$ summand to $N$ by $\phi_g^{-1} \circ F_g(f)$. It is a short calculation to check that this is also sufficient to define a morphism of $C^G$. The construction in the other direction is more obvious: given a map $h : \Sigma(M) \rightarrow (N, \phi)$, restrict $h$ to the $M$ summand of $\Sigma(M)$. This describes a natural isomorphism $\text{Hom}_C(-, \Phi(-)) \cong \text{Hom}_{C^G}(\Sigma(-), \cdot)$.

Now let $f$ be a map $N \rightarrow M$; I need to produce a map $(N, \phi) \rightarrow \Sigma(M)$. The formula is similar to the above construction - $F_g(f) \circ \phi_g$ is a map $N \rightarrow F_g(M)$, and the product of these as $g$ ranges over $G$ is the desired map $(N, \phi) \rightarrow \Sigma(M)$. Given a map $(N, \phi) \rightarrow \Sigma(M)$ we can look at its projection to the factor $M$ of $\Sigma(M)$ to get back a map $N \rightarrow M$ of $C$. \hfill \Box

2 Duality

In this section I will assume that $G$ is abelian. For certain results I also require that the natural map from $G$ to $\tilde{G} = \text{Hom}(\text{Hom}(G, k^*), k^*)$ is an isomorphism.\footnote{Note that this implies $\text{char}(k) \nmid |G|$.} Recall that $\tilde{G}$ acts on the category $C^{\tilde{G}}$. During this section, I will consider the category $C^{G}$ as well as $(C^{G})^{\tilde{G}}$. In order to avoid confusion, let $\Phi_1, \Phi_2$ be the forgetful functors $C^{G} \rightarrow C$ and $(C^{G})^{\tilde{G}} \rightarrow C^{G}$, respectively.

Now I am set up to prove the duality phenomenon I am interested in, but first a motivating example:

Example 12. Consider the category of sheaves of $k$ vector spaces on $G$. This is a simple enough category to understand, and the action of $G$ is clear. In fact, we can identify $(\text{Sh}_G)^{G} \cong \text{Vec}_k$. This equivalence can be built in one direction by restricting to $0 \in G$. Under this equivalence, the $\tilde{G}$ action becomes the trivial one on $\text{Vec}_k$. Now consider taking equivariants again: $\text{Vec}_{k}^{\tilde{G}}$. Since the action of $\tilde{G}$ was trivial, this is the category of $k$-linear representations of $\tilde{G}$. What is more, this category has an action of $\tilde{G} \equiv G$ which simply twists a $\tilde{G}$ representation by a character of $\tilde{G}$. Provided that the characteristic of $k$ does not divide the order of $\tilde{G}$, we know there is an equivalence between $k$ representations of $\tilde{G}$ and sheaves of $k$ vector spaces on $G$. Under this equivalence, the action of $G$ becomes translation of sheaves by the regular action of $G$ on itself. Thus we see that after taking equivariants twice, we return to the same category that we began with and to the same action of $G$.

The generalization of the above phenomenon is the goal of this section:
Proposition 13. Let $C$, $G$, and the action of $G$ on $C$ be as previously defined. The natural averaging functor $\Sigma : C \rightarrow C^G$ factors through the forgetful functor $\Phi_2 : (C^G)^\hat{G} \rightarrow C^G$. Call this functor $\overline{\Sigma}$. Here is a diagram of the situation:

![Diagram of the situation](image)

The functor $\overline{\Sigma}$ is always faithful. In cases where $G \equiv \hat{G}$, the functor $\overline{\Sigma}$ is fully faithful. When, further, all projectors in $C$ split, $\overline{\Sigma}$ is essentially surjective, thus an equivalence.

Proof. I first need to demonstrate that $\Sigma$ factors through $\Phi_2$. For an arbitrary $A \in C$, consider the $\hat{G}$ twists of the $G$-equivariant structure on $\Sigma(A)$. The natural equivariant structure on $\Sigma(A)$ is

$$\bigoplus_{x \in G} [(\Delta_{x,g}(A))^{-1} : F_{xg}(A) \rightarrow F_{g}(F_x(A))]$$

and for $\chi \in \hat{G}$ the $\chi$ twist is

$$\bigoplus_{x \in G} [\chi(g)\Delta_{x,g}(A))^{-1} : F_{xg}(A) \rightarrow F_{g}(F_x(A))]$$

Call these structures $\phi_\bullet$ and $\phi'_\bullet$ respectively. I must construct an isomorphism of $C^G$ from $\Sigma(A)$ to $F_\chi(\Sigma(A))$. I define

$$\lambda_\chi = \bigoplus_{x \in G} \left[(\chi(x))^{-1} \cdot \text{id} : F_x(A) \rightarrow F_x(A)\right]$$

and this is such an isomorphism. Furthermore, the morphisms are multiplicative in the sense of (2) for the group $\hat{G}$ so $(\Sigma(A), \lambda_\bullet)$ is an object of the category $(C^G)^\hat{G}$. Now given a morphism $f : A \rightarrow B$ of the category $C$, I should check that $\Sigma(f)$ is a morphism of the $\hat{G}$ equivariant structures on $\Sigma(A)$ and $\Sigma(B)$. On the one hand

$$\lambda_\chi \circ \Sigma(f) = \bigoplus_{x \in G} [\chi(x)^{-1}\text{id}_{F_x(B)} \circ F_x(f)]$$

and on the other

$$F_\chi(\Sigma(f)) \circ \lambda_\chi = \bigoplus_{x \in G} [F_\chi(F_x(f) \circ \chi(x)^{-1}\text{id}_{F_x(A)}]]$$
These are equal, so I have shown how to construct the desired functor $\Sigma$.

It is immediate that $\Sigma$ is faithful. To see that $\Sigma$ is full, let $f = (f_{ij}) : \Sigma(A) \to \Sigma(B)$ be a morphism. In this notation $f_{ij}$ is a morphism $F_j(A) \to F_i(B)$. Since $f$ is a morphism of $G$ equivariant structures

$$f_{ig,jg} = \Delta_{i,g}(B) \circ F_g(f_{i,j}) \circ \Delta_{j,g}(A)^{-1}$$

This means that $f$ is determined by the morphisms $f_{x,1}$, where $x$ runs over $G$. Since $f$ is also a morphism of $\hat{G}$ equivariant structures, it must satisfy $f_{h,1} = \chi(h)^{-1}f_{h,1}$ for all $\chi$. Now the only $h \in G$ which satisfies $\chi(h) = 1$ for all $\chi$ is 1 itself; here I used that $G$ is isomorphic to its double dual. Thus, if $h \neq 1$, we must have $f_{h,1} = 0$. This means that

$$f = \bigoplus_{x \in G} \left[ F_x(f_{1,1}) : F_x(A) \to F_x(B) \right]$$

so $f$ is in the image of $\Sigma$. This is as much as I can say about $\Sigma$ without assuming anything further about the category $C$.

Now I assume that in $C$ all projectors (idempotent endomorphisms) split. A projector $p \in \text{End}(M)$ is said to split if $M \cong M_0 \oplus M_1$ so that $p|M_0 = 0$ and $p|M_1$ is the inclusion of $M_1$ into $M_0 \oplus M_1$. Such a decomposition is unique since it is determined by universal property. To prove essential surjectivity, the idea is the following: the summand $A$ of $\Sigma(A)$ is the image of a certain projector. Namely the average over $\hat{G}$ of the equivariant structures $\lambda_\chi : \Sigma(A) \to F_\chi(\Sigma(A))$. In $C^G$, the $\lambda_\chi$ are not endomorphisms of $\Sigma(A)$, because $F_\chi(\Sigma(A))$ is a different (albeit isomorphic) object of $C^G$. However, upon applying the forgetful functor $\Phi_1$ back down to $C$, these all become well defined endomorphisms of $[\Sigma(A)]$. In fact, the average of the $\lambda_\chi$ is easy to compute. On the $F_\chi(A)$ summand, $\phi_\chi$ acts as multiplication by $\chi(x)^{-1}$. Hence the average of the $\phi_\chi$ is 0 on all but the identity summand of $\Sigma(A)$ and is 1 on the identity summand. Thus $A$ is the image of the projector

$$\frac{1}{|G|} \sum_{\chi \in \hat{G}} \Phi_1(\lambda_\chi)$$

in the object $[\Sigma(A)]$.

Fix an object $M \in (C^G)^{\hat{G}}$ with $G$-equivariant structure $\phi_\bullet$ and $\hat{G}$-equivariant structure $\lambda_\bullet$. The $\lambda_\bullet$ are $C$ endomorphisms of $M$, as explained above and furthermore they are multiplicative. That the $\lambda_\bullet$ are honestly multiplicative uses two facts about the $\hat{G}$-equivariant structure on $C^G$: that the isomorphisms $\Delta_{\bullet,\bullet}$ are always identities and that the functors $F_\chi$
act trivially on morphisms. Hence $M$ has the structure of a $\tilde{G}$ representation in $C$. In fact, the composite functor $\Phi_1 \Phi_2$ factors through the forgetful functor from the category of $\tilde{G}$ representations in $C$ to $C$. Since projectors split in $C$, the $\tilde{G}$ module $M$ can be decomposed as a direct sum of isotypic components for irreducible representations of $\tilde{G}$:

$$M \cong \bigoplus_{g \in G} M_g$$

I claim that $\Sigma(M_0) \cong M$ as objects of $(C^G)^{\tilde{G}}$. Let $i$ be the inclusion of $M_0$ into $M$. The adjunction $\Sigma \dashv \Phi_1$ gives me a map $i' : \Sigma(M_0) \to M$. I claim that this is a morphism not just in $C^G$ but in $(C^G)^{\tilde{G}}$ and that it is an isomorphism.

Now $M$ is decomposed as a direct sum of characters $g \in G$ of $\tilde{G}$. Fix $x \in G$ and consider the $M_x$ summand of $M$. Since the $F_g$ were additive functors to start with, this corresponds to a summand $F_g(M_x)$ of $F_g(M)$. Direct calculation shows that this summand is the image under $\phi_g$ of the summand of $M$ on which $\lambda_x$ acts by $xg^{-1}$. Since $\phi_g$ is an isomorphism, this shows that $\phi_g$ maps $M_{xg^{-1}}$ isomorphically to $F_g(M_x)$. For all $g \in G$, $F_g(N)$ is a summand of $\Sigma(N)$ and is mapped isomorphically by $\phi_g^{-1} \circ F_g(i)$ to $M_{g^{-1}}$. This shows that the map $i'$ is an isomorphism in the category $C^G$ and I just need to verify that $i'$ is a morphism in the category $(C^G)^{\tilde{G}}$. But recall that in the construction of the functor $\Sigma$, I gave $\Sigma(N)$ a $\tilde{G}$ equivariant structure which scales by $\chi(g)^{-1}$ on the $F_g(N)$ summand. This shows that $i'$ respects the $\tilde{G}$ equivariant structure on $M$ and $\Sigma(M_0)$ so $i'$ is a $(C^G)^{\tilde{G}}$ morphism. Hence $\Sigma$ is essentially surjective.

3 Equivariant Derived Categories

When $\mathcal{A}$ is an abelian category with $G$ action by exact functors $F_\bullet$, the derived category $D(\mathcal{A})$ has a $G$ action also. Then one may wish to consider the category $D(\mathcal{A})^G$. On the other hand, $\mathcal{A}^G$ is an abelian category so $D(\mathcal{A}^G)$ is also a natural construction, and there is a natural functor $i : D(\mathcal{A}^G) \to D(\mathcal{A})^G$. In this section, I will focus on understading something about the functor $i$. Indeed, it is not clear a priori that $i$ is full, or faithful, or essentially surjective! The issue with fullness and faithfulness is that, because of the way morphisms are defined in $D(\mathcal{A})^G$, a morphism between objects with $G$ equivariant structure need only commute with the equivariance maps up to homotopy, and this chain homotopy need not
itself respect the equivariant structures in any direct way. Essential surjectivity fails for a similar reason: objects of $D(A)^G$ need only have $G$ equivariant structures multiplicative up to chain homotopy\(^2\). Indeed, this is a question of efficacy of descent for the derived category; such descent is known to fail in many natural cases.

In this section I will assume that the category $A$ is $k$-linear and has enough injectives. During the argument to follow, I will want $D(A^G)$ to satisfy the hypotheses of the following result due to Bökstedt and Neeman [BN93]:

**Proposition 14** (Bökstedt-Neeman). Let $\mathcal{I}$ be a triangulated category with direct sums. Suppose $e : X \to X$ is an idempotent in $\mathcal{I}$. Then $e$ is split in $\mathcal{I}$.

In order to use this fact, I need to require that $A$ be closed under infinite direct sums.

First I will prove the following very useful lemma:

**Lemma 15.** An object $M$ of $A^G$ is injective iff it is injective as an object of $A$.

\textit{Proof.} Suppose $(M, \phi)$ is injective as an object of $A$. Then for any injection $i : W \hookrightarrow Z$, $i^*$ is a surjection $\text{Hom}_A(Z, M) \to \text{Hom}_A(W, M)$. Now suppose that $Z$ and $W$ have their own $G$ equivariant structures and that $i$ is a morphism of $A^G$. Then there is a map $i^* : \text{Hom}_{A^G}(Z, M) \to \text{Hom}_{A^G}(W, M)$ and I need to show that it is a surjection. This is part of the long exact sequence for the sequence of $G$-modules

$$0 \to \text{Hom}_A(Z/W, M) \to \text{Hom}_A(Z, M) \to \text{Hom}_A(W, M) \to 0$$

Since the characteristic of $k$ doesn’t divide the order of $G$, the operation of taking $G$ invariants is exact. In particular I have the surjection I needed and $M$ is injective as an object of $A^G$.

Now suppose that $(M, \phi)$ is injective in $A^G$ and let $i : W \hookrightarrow Z$ and $f : W \to M$ be morphisms of $A$. The following is a diagram in the category $A$:

$$
\begin{array}{ccc}
\Sigma(Z) & \longrightarrow & Z \\
\downarrow & & \downarrow i \\
\Sigma(W) & \longrightarrow & W \longrightarrow M \\
\end{array}
$$

\(^2\)We will only have to consider the homotopy category of the chain category, because we will assume $A$ has enough injectives.
In order to extend \( f \) to \( Z \), it is enough to extend \( f \) to an \( \mathcal{A}^G \) morphism from \( \Sigma(W) \). Then by injectivity of \( M \) in \( \mathcal{A}^G \), the morphism will extend to one from \( \Sigma(Z) \) which may be restricted to \( Z \) in \( \mathcal{A} \). Such an extension exists by the adjunction \( \Sigma \dashv \Phi \). \( \square \)

It is a standard result that injective resolutions can be used to compute Hom spaces in derived categories\(^3\). Now let \( M^\bullet, N^\bullet \) be objects of \( D^b(\mathcal{A}^G) \), and let \( I^\bullet \) be an injective resolution of \( N^\bullet \) in \( \mathcal{A}^G \).

\[
\text{Hom}_{D(\mathcal{A}^G)}(M^\bullet, N^\bullet) \cong H^0(\text{RHom}_{\mathcal{A}^G}(M^\bullet, I^\bullet))
\]

\[
= H^0\left( \bigoplus_{i,j} \text{Hom}_{\mathcal{A}^G}(M^i, I^j) \right)
\]

\[
= H^0\left( \bigoplus_{i,j} \text{Hom}_{\mathcal{A}}(M^i, I^j)^G \right)
\]

\[
= H^0(\text{RHom}_{\mathcal{A}}(M^\bullet, I^\bullet)^G)
\]

Let \( M^\bullet, N^\bullet, I^\bullet \) be as above and consider these as objects of \( D(\mathcal{A})^G \) via the functor \( i \). From (4) I know that

\[
\text{Hom}_{D(\mathcal{A})^G}(M^\bullet, N^\bullet) \cong \text{Hom}_{D(\mathcal{A})}(M^\bullet, N^\bullet)^G
\]

But note also that this is computed by the injective resolution \( I^\bullet \):

\[
(H^0(\text{RHom}_{\mathcal{A}}(M^\bullet, I^\bullet)))^G
\]

Taking \( G \) invariants is left exact, and this property is why there is a natural map:

\[
H^0\left( \text{RHom}_{\mathcal{A}}(M^\bullet, I^\bullet)^G \right) \rightarrow (H^0(\text{RHom}_{\mathcal{A}}(M^\bullet, I^\bullet)))^G
\]

But again, taking \( G \) invariants is actually exact so the above map is an isomorphism. I have just proved the following:

**Lemma 16.** Let \( \mathcal{A} \) be as above: with enough injectives and \( k \)-linear. Then the natural functor \( i : D^*(\mathcal{A}^G) \rightarrow D^*(\mathcal{A})^G \) is fully faithful when * is ‘+’ or ‘b’.

I would now like to prove that that \( i \) is essentially surjective. Suppose that \( (M, \phi_\bullet) \) is an object of \( D^+(\mathcal{A}) \) together with a ‘sloppy’ \( G \)-equivariant structure. I would like to show that

\(^3\)See for example the consequences of Proposition 2.56 in [Huy06].
$M$ is $D^+(\mathcal{A})^G$ isomorphic to an object with a strict $G$-equivariant structure. Lemma (11) gives me a sequence in $D(\mathcal{A}^G)$:

$$(M, \phi_\ast) \longrightarrow \Sigma(M) \longrightarrow (M, \phi_\ast)$$

such that the composition is the multiplication by $|G|$ map. Dividing the second of these morphisms by $|G|$, $(M, \phi_\ast)$ is the image of a projector, call it $e$, of $\Sigma(M)$. By ([BN93], 3.2), the projector $e$ is split in $D(\mathcal{A}^G)$. Say $T$ is the image of the projector. Then it is an easy check to show that $T$ also satisfies the universal property for an image of $e$ in $D(\mathcal{A})^G$, so $T$ and $M$ are $D(\mathcal{A})^G$ isomorphic. I have now shown:

**Lemma 17.** Let $\mathcal{A}$ be a $k$-linear abelian category, closed under infinite direct sums, and with enough injectives. Let $G$ act on $\mathcal{A}$ by exact equivalences and let $G \cong \text{Hom}(\text{Hom}(G, k^\ast), k^\ast)$. Then the natural functor $i : D^+(\mathcal{A}^G) \to D^+(\mathcal{A})^G$ is an equivalence of categories when $*$ is ‘+’ or ‘b’.

**References**
