Math 234 Review Problems for the Final Exam

Marc Conrad

December 13, 2007

Directions: Answer each of the following questions. Pages 1 and 2 contain the problems. The solutions are on pages 3 through 7.

Problem 1. Solve each of the following integration problems. While solving each one, try to determine the best (i.e., fastest) way to calculate the given integral. Some may require that you calculate directly, while others will be faster to compute using integration theorems.

(a) Let $\mathbf{F} = \frac{1}{3}x^3z\mathbf{i} + (x+z)\mathbf{j} + \frac{1}{2}y^2z^2\mathbf{k}$. Find the flux of $\mathbf{F}$ *inward* across the surface S that consists of the portion of the cone $z = \sqrt{x^2+y^2}$ for $1 \leq z \leq 2$ together with its "caps" $x^2+y^2 \leq 1$, $z = 1$ and $x^2+y^2 \leq 4$, $z = 2$.

(b) Integrate the function $f(x,y,z) = z^2$ over the unit sphere $x^2+y^2+z^2 = 1$.

(c) Let C be the curve that consists of the intersection of the plane $2x+y+z = 5$ with the "square tube" consisting of the planes $x = 0$, $x = 1$, $y = 0$ and $y = 1$, oriented counterclockwise as viewed from above. Find the circulation of the vector field $\mathbf{F} = -y\mathbf{i} + \frac{1}{2}z^2\mathbf{j} + 2\mathbf{k}$ along C.

(d) Find the flux of the vector field $\mathbf{F} = (y^2+\ln(y^2+z^2))\mathbf{i} + \sqrt{x^2+\sqrt{z^4+1}}\mathbf{j} + (e^{x\cos y^2}+2z)\mathbf{k}$ outward across the sphere $x^2+y^2+z^2 = a^2$.

(e) Find the flux of the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ in the direction towards from the origin across the portion of the paraboloid $z = 1 - x^2 - y^2$ in the first octant.

Problem 2. Which of the following make sense? Of the ones that make sense, which are always equal to zero for "nice" functions f or vector fields $\mathbf{F}$?

$$\nabla \times (\nabla f)$$
$$\nabla \cdot (\nabla f)$$
$$\nabla \times (\nabla \cdot \mathbf{F})$$
$$\nabla \cdot (\nabla \times \mathbf{F})$$

Problem 3. State each of the following. Be sure to include the necessary hypotheses.

(a) Stokes' Theorem

(b) The Divergence Theorem

(c) The Cauchy-Riemann equations

Problem 4. Let $f(x+iy) = 2x^2 - 2y^2 + 4xyi$. Find the derivative of f.

Problem 5. For $x > 0$, define $f(x+iy) = \ln\left((x^2+y^2)^{1/2}\right) + \arctan\left(\frac{y}{x}\right) i$.

(a) Is f differentiable?

(b) Rewrite f as a function of the polar coordinates r and θ.

(c) Now let α be a *fixed* complex number and define the function $g_\alpha(z) = e^{\alpha f(z)}$. Use the polar form of f that you found in part (b) to show that if n is a natural number, then $g_n(z) = z^n$.

Recall that the natural numbers are $1, 2, 3, \ldots$

Solutions

1a. We are asked for the flux across a closed surface, so we can apply the Divergence Theorem. Furthermore, since the surface consists of three parts, the Divergence Theorem will allow us to do fewer calculations. So we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}_{\text{in}}\, d\sigma = -\iiint_D \nabla \cdot \mathbf{F}\, dx\, dy\, dz$$

(Don't forget the minus sign to account for the fact that $\mathbf{n}_{\text{in}}$ points in instead of out!)

We can easily calculate that $\nabla \cdot \mathbf{F} = (x^2 + y^2)z$. The integration will be simplest in cylindrical coordinates. Then $\nabla \cdot \mathbf{F} = r^2 z$ and we're left with

$$\iint_S \mathbf{F} \cdot \mathbf{n}_{\text{in}}\, d\sigma = -\int_{z=1}^{2} \int_{\theta=0}^{2\pi} \int_{r=0}^{z} r^3 z\, dr\, d\theta\, dz$$

(We have r^3 instead of r^2 because of the streching factor r that comes from integration in cylindrical coordinates).

If we integrate this, we will get an answer of $-\frac{21\pi}{4}$.

1b. Even though we have a closed surface, the Divergence Theorem doesn't apply because it only works for integrating vector fields over surfaces. It cannot help us integrate a scalar valued function over a surface. We can use spherical coordinates (keeping in mind that $\rho = 1$ on the surface of the unit sphere) to parametrize the sphere and obtain

$$\mathbf{r}(\theta, \phi) = \sin\phi \cos\theta \mathbf{i} + \sin\phi \sin\theta \mathbf{j} + \cos\phi \mathbf{k}$$
$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi$$

Now the integral of f over the sphere is

$$\iint_S f(x, y, z)\, d\sigma = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} f(x(\theta, \phi), y(\theta, \phi), z(\theta, \phi)) |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, d\phi\, d\theta$$

Direct computation yields $\mathbf{r}_\phi = \cos\phi \cos\theta \mathbf{i} + \cos\phi \sin\theta \mathbf{j} - \sin\phi \mathbf{k}$ and $\mathbf{r}_\theta = -\sin\phi \sin\theta \mathbf{i} + \sin\phi \cos\theta \mathbf{j}$. The cross product is then $\sin^2\phi \cos\theta \mathbf{i} + \sin^2\phi \sin\theta \mathbf{j} +$

$\sin\phi\cos\phi\mathbf{k}$. If we calculate the length of this, we get $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi$ (You'll need to use the identity $\sin^2 x + \cos^2 x = 1$ a couple of times).
Now $f(x(\theta,\phi), y(\theta,\phi), z(\theta,\phi)) = (z(\theta,\phi))^2 = \cos^2\phi$. So were left with

$$\iint_S f(x,y,z)\,d\sigma = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \cos^2\phi\sin\phi\,d\phi\,d\theta$$

Computing this integral will yield a result of $\frac{4\pi}{3}$.

1c. C is a closed curve, so we can apply Stokes' Theorem. Furthermore, C consists of four line segments, so using the theorem will cut down on the number of calculations we have to make. The surface S bounded by C is the portion of the plane $2x + y + z = 5$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$. So S is the graph of the function $f(x,y) = 5 - 2x - y$. Then we have that

$$\iint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}_{\text{up}}\,d\sigma$$

(We have used the right hand rule to determine that we want $\mathbf{n}_{\text{up}}$ instead of $\mathbf{n}_{\text{down}}$).

Now $\nabla\times\mathbf{F} = -z\mathbf{i} + \mathbf{k}$ and so

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}_{\text{up}}\,d\sigma = \iint_R (-z\mathbf{i}+\mathbf{k})\cdot(-f_x, -f_y, 1)\,dx\,dy$$

R is the shadow of S in the xy plane, so in this case R is the square $0 \le x \le 1$, $0 \le y \le 1$. We also have that $f_x = -2$ and $f_y = -1$ and so we're left with

$$\begin{aligned}\int_{x=0}^{1}\int_{y=0}^{1} -2z + 1\,dy\,dx &= \int_{x=0}^{1}\int_{y=0}^{1} -2(5-2x-y) + 1\,dy\,dx \\ &= \int_{x=0}^{1}\int_{y=0}^{1} (4x+2y-9)\,dy\,dx\end{aligned}$$

Computing the integral will yield an answer of -6.

1d. We are asked for the flux across a closed surface, so we can apply the Divergence Theorem. In this case, although the surface isn't particularily bad, the vector field is, so the theorem will help us a lot. In fact, $\nabla \cdot \mathbf{F} = 2$ and so we have that

$$\iint_S \mathbf{F} \cdot \mathbf{n}_{\text{out}}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dx\, dy\, dz = \iiint_D 2\, dx\, dy\, dz$$

So the answer will simply be 2 times the volume of the sphere, or $\frac{8}{3}\pi a^3$.

1e. We are asked for the flux across a surface that is *not* closed, and so the Divergence Theorem won't be of much use. So we are left with directly calculating the flux. S is the graph of the function $f(x, y) = 1 - x^2 - y^2$ and we have that the shadow of S in the xy plane is the quarter of the circle $x^2 + y^2 \leq 1$ that lies in the first quadrant. We also note that the normal that points towards the origin in this case is $\mathbf{n}_{\text{down}}$ and so we have

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}_{\text{down}}\, d\sigma &= \iint_R \overrightarrow{(y, -x, z)} \cdot \overrightarrow{(f_x, f_y, -1)}\, dx\, dy \\
&= \iint_R \overrightarrow{(y, -x, z)} \cdot \overrightarrow{(-2x, -2y, -1)}\, dx\, dy \\
&= \iint_R -z\, dx\, dy = \iint_R (x^2 + y^2 - 1)\, dx\, dy
\end{aligned}$$

where R is the shadow of S in the xy plane and we have replaced z with $f(x, y)$ in the last step.
This integral is best done in polar coordinates, where we will have

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}_{\text{down}}\, d\sigma &= \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} (r^2 - 1) r\, d\theta\, dr \\
&= \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} r^3 - r\, d\theta\, dr
\end{aligned}$$

Calculating this integral will give us $-\frac{\pi}{8}$.

2. The curl of the gradient of a scalar valued function f is $\nabla \times (\nabla f) = (f_{zy} - f_{yz})\mathbf{i} + (f_{xz} - f_{zx})\mathbf{j} + (f_{yx} - f_{xy}) = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, so it is always zero.

The divergence of the gradient of a scalar valued function f is $\nabla \cdot (\nabla f) = f_{xx} + f_{yy} + f_{zz}$. This makes sense, and in fact it is the *Laplacian* of f, denoted

Δf or $\nabla^2 f$. Its value will depend on the specific function given, so it is not necessarily zero.

The divergence of a vector field $\mathbf{F}$ is a scalar valued function, and the curl only operates on vector fields, not on functions, so the curl of the divergence $\nabla \times (\nabla \cdot \mathbf{F})$ doesn't make sense.

The divergence of the curl of a vector field $\mathbf{F} = M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$ is $\nabla \cdot (\nabla \times \mathbf{F}) = P_{yx} - N_{zx} + M_{zy} - P_{xy} + N_{xz} - M_{yz} = 0$, so it is always zero.

3a. If the curve C is the boundary of the surface S then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

where the orientations of C and S are related by the right hand rule.

3b. If the surface S is the boundary of the region D then

$$\iint_S \mathbf{F} \cdot \mathbf{n}_{\text{out}} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dx \, dy \, dz$$

where $\mathbf{n}_{\text{out}}$ is the outward unit normal to S.

3c. If the function $f(x+iy) = u(x,y) + v(x,y)i$ is complex differentiable then the partial derivatives of u and v satisfy the relations

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

4. We can proceed in one of two ways. The first method is to use the rule that $f'(x+iy) = u_x + v_x i$. Here $u(x,y) = 2x^2 - 2y^2$ and $v(x,y) = 4xy$, so $u_x = 4x$ and $v_x = 4y$. So $f'(x+iy) = 4x + 4yi$.

The second method is to note that $z^2 = (x+yi)^2 = x^2-y^2+2xyi$ and therefore $f(z) = 2z^2$. Then by the power rule $f'(z) = 4z$. (Note that $4z = 4x + 4yi$, so indeed both methods yield the same answer).

5a. Yes. We have $u(x,y) = \ln\left((x^2+y^2)^{1/2}\right)$ and $v(x,y) = \arctan\left(\frac{y}{x}\right)$. Then we have that

$$
\begin{aligned}
u_x &= \frac{x}{x^2+y^2} \\
u_y &= \frac{y}{x^2+y^2} \\
v_x &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} \\
v_y &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{1}{x+\frac{y^2}{x}}\left(\frac{x}{x}\right) = \frac{x}{x^2+y^2}
\end{aligned}
$$

So $u_x = v_y$ and $u_y = -v_x$. So the Cauchy-Riemann equations are satisfied and therefore f is complex differentiable.

5b. We use the relations $r^2 = x^2+y^2$ and $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$ (we can also work out that $\frac{y}{x} = \tan\theta$ by drawing a picture of the complex number $z = x+yi$ in the plane and using geometry). This gives us that $\ln\left((x^2+y^2)^{1/2}\right) = \ln r$ and $\arctan\left(\dfrac{y}{x}\right) = \arctan(\tan\theta) = \theta$. Therefore we have that

$$f(r\cos\theta + r\sin\theta i) = \ln r + \theta i$$

5c. We have that

$$
\begin{aligned}
g_n(z) &= e^{nf(z)} = e^{n\ln r + n\theta i} \\
&= e^{\ln r^n + n\theta i} = e^{\ln r^n} e^{n\theta i} \\
&= r^n e^{n\theta i} = r^n(\cos(n\theta) + \sin(n\theta)i)
\end{aligned}
$$

But we have as a consequence of Proposition 2 from handout C1 that $z^n = r^n(\cos(n\theta)+\sin(n\theta)i)$. Therefore $g_n(z) = z^n$.