Finding Potential Functions

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1 Introduction

Given a vector field $\mathbf{F}$, one thing we may be asked is to find a potential function for $\mathbf{F}$. That is, we want to find a scalar-valued function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$. In general, we cannot guarantee the existance of such a function. To establish the existance of a potential for $\mathbf{F}$ (that is, to show that $\mathbf{F}$ is *conservative*), we can use the Component Test given on Page 1164 of the text. If it is the case that $\mathbf{F}$ is conservative, then we can find the potential f through a systematic procedure that is best illustrated by example.

2 An Example

2.1 Setting up the problem

Let $\mathbf{F}$ be the vector field $2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2z)\mathbf{k}$. Find a potential function for $\mathbf{F}$.

One can use the component test to show that $\mathbf{F}$ is conservative, but we will skip that step and go directly to finding the potential. We want to find f such that $\nabla f = \mathbf{F}$. That is we want to have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = 2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2z)\mathbf{k}$$

So the problem is to find a function f such that

$$\frac{\partial f}{\partial x} = 2xy \quad (1)$$

$$\frac{\partial f}{\partial y} = x^2 + 2yz \quad (2)$$

$$\frac{\partial f}{\partial z} = y^2 + 2z \quad (3)$$

2.2 Step 1: Finding a preliminary form for f

We have that $\frac{\partial f}{\partial x} = 2xy$. Therefore f is given by the indefinite integral $f(x,y,z) = \int 2xy\,dx$. Solving this yields

$$f(x,y,z) = \int 2xy\,dx = x^2y + C(y,z) \quad (4)$$

Note that the book uses the notation $g(y,z)$ instead of $C(y,z)$. Now, before proceeding any further, we explore why the term $C(y,z)$ appears in our expression, and in particular why we must allow it to depend on y and z.

2.3 The reason for the term $C(y,z)$

When we calculate a single variable indefinite integral, we need a constant of integration C. For example $\int x\,dx = \frac{1}{2}x^2 + C$. The reason for the C term is that we want the *most general* antiderivative we can find. The function $f_1(x) = \frac{1}{2}x^2$ has the property that $f_1'(x) = x$, but so do the functions $f_2(x) = \frac{1}{2}x^2 + 5$, $f_3(x) = \frac{1}{2}x^2 - \pi$ and, in general $f(x) = \frac{1}{2}x^2 + C$ for any constant C.

Similarily, in our case we want (for the time being) to find the most general function $f(x,y,z)$ that has the property that $\frac{\partial f}{\partial x} = 2xy$. The function $f_1(x,y,z) = x^2y$ has the property that $\frac{\partial f_1}{\partial x} = 2xy$, but so each of the following functions

$$\begin{aligned} f_2(x,y,z) &= x^2y - 2 \\ f_3(x,y,z) &= x^2y + yz \\ f_4(x,y,z) &= x^2y + \sin ye^{2z-\cos(yz)} + \tan^{-1}\left(\frac{yz}{y^2+z^2+1}\right) \end{aligned}$$

In general $f(x,y,z) = x^2y + C(y,z)$ will have the desired property for any function $C(y,z)$. Hence, the *most general* form of f with the property that $\frac{\partial f}{\partial x} = 2xy$ is $f(x,y,z) = x^2y + C(y,z)$ and it is therefore the indefinite integral.
We now return to the problem of finding the potential.

2.4 Step 2: Refining f

We have found an f (actually an infinite family of f's) such that $\frac{\partial f}{\partial x} = 2xy$. But we also have conditions on $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$. We first work with our condition for $\frac{\partial f}{\partial y}$. On the one hand, by Equation (2) we have that

$$\frac{\partial f}{\partial y} = x^2 + 2yz$$

On the other hand, taking the partial derivative of both sides of Equation (4) with respect to y gives us that

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial C}{\partial y} \tag{5}$$

Taken together, Equations (2) and (5) imply that $x^2 + 2yz = x^2 + \frac{\partial C}{\partial y}$ and therefore

$$\frac{\partial C}{\partial y} = 2yz$$

Integrating this result gives us that

$$C(y,z) = \int 2yz\,dy = y^2z + C_1(z) \tag{6}$$

The reason for having $C_1(z)$ in Equation (6) is the same as the reason for having $C(y,z)$ in Equation (4). That is, $C(y,z) = y^2z + C_1(z)$ has the property that $\frac{\partial C}{\partial y} = 2yz$ for any choice of $C_1(z)$ and (at the moment) we want the most general expression for $C(y,z)$ that we can find.

Now, substituting our expression for $C(y,z)$ from Equation (6) into Equation (4) gives us that

$$f(x,y,z) = x^2y + y^2z + C_1(z) \tag{7}$$

2.5 Step 3: Obtaining the final answer

The expression we have obtaining for f in Equation (7) has the property that $\frac{\partial f}{\partial x} = 2xy$ and $\frac{\partial f}{\partial y} = x^2 + 2yz$. Now we use our condition on $\frac{\partial f}{\partial z}$, proceeding in a similar manner as in Step 2. On the one hand, Equation (3) tells us that

$$\frac{\partial f}{\partial z} = y^2 + 2z$$

On the other hand, taking the partial derivative of both sides of Equation (7) gives us that

$$\frac{\partial f}{\partial z} = y^2 + \frac{dC_1}{dz} \tag{8}$$

Taken together, Equations (3) and (8) imply that $y^2 + 2z = y^2 + \frac{dC_1}{dz}$ and therefore

$$\frac{dC_1}{dz} = 2z$$

Integrating the result gives us that

$$C_1(z) = \int 2z\,dz = z^2 + C_2 \tag{9}$$

Where C_2 is an arbitrary constant. Substituting our expression for $C_1(z)$ from Equation (9) into Equation (7) gives us that

$$f(x,y,z) = x^2y + y^2z + z^2 + C_2 \tag{10}$$

We can now verify by direct calculation that

$$\nabla f = 2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2z)\mathbf{k} = \mathbf{F}$$

Therefore f as given in Equation (10) is a potential function for any choice of the constant C_2. That is, we can let C_2 be any (fixed) real number we want. In practice it is often convenient for calculations to let $C_2 = 0$.