Math 340 Solutions to Review Problems for Exam 1

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September 28, 2007

1 Problems from the book

Section 1.1: 18

Yes, because a homogeneous system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

always has the trivial solution $x_1 = x_2 = \ldots = x_n = 0$.

Section 1.2: 2

For part (a) the matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

For part (b) the matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Section 1.4: 27

To prove property (b), proceed as follows: Let $A = (a_{ij}), B = (b_{ij})$ and $C = A + B$.
Then $c_{ij}^T = (a_{ij} + b_{ij})^T = (a_{ji} + b_{ji}) = a_{ji} + b_{ji} = a_{ij}^T + b_{ij}^T$.
So indeed $(A + B)^T = A^T + B^T$.

To prove property (d), proceed as follows. Let $A = (a_{ij})$ and $D = rA$.
Then $d_{ij}^T = (ra_{ij})^T = ra_{ji} = r(a_{ij}^T)$.
So indeed $(rA)^T = r(A^T)$.

The rest of the solutions for this section can be found in the book.

2 Definitions

1. The equation $E[u] = g$ is *linear* if it satisfies the following two conditions

(1) Given two unknowns u_1 and u_2, we have $E[u_1 + u_2] = E[u_1] + E[u_2]$.

(2) Given an unknown u and a scalar (i.e. a number) λ, we have that $E[\lambda u] = \lambda E[u]$.

2. A system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

is *consistent* if it has at least one solution, i.e., if we can find an n-tuple of numbers $(c_1, c_2, \ldots, c_n)$ which satisfies the equation. It is *inconsistent* if it has no solution.

3. A system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

is *homogeneous* if $b_1 = b_2 = \ldots = b_m = 0$. It is *inhomogeneous* if there is at least one non-zero b_i.

4. A solution $(c_1, c_2, \ldots, c_n)$ to a homogeneous system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

is *non-trivial* if there is at least one non-zero c_i.

5. An $n \times n$ matrix A is invertible if there is an $n \times n$ matrix B so that $AB = BA = I_n$, where I_n is the $n \times n$ identity matrix.

6. The three elementary row operations are

(1) Interchanging two rows.

(2) Mulitplying a row by a *non-zero* constant. (Don't forget the "non-zero"!)

(3) Adding a constant multiple of one row to a different row.

3 Additional Problem

We need to do two things here. First, we need to show that the matrix

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

has the property that $AB = BA$ for *all* 2×2 matrices B. Then we need to show that if A has the property that $AB = BA$ for *all* 2×2 matrices B, then it can be written in the above form.

First we will assume that

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

Then letting B be an arbitrary 2×2 matrix

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

we can readily compute that AB and BA are both equal to

$$\begin{bmatrix} a(b_{11}) & a(b_{12}) \\ a(b_{21}) & a(b_{22}) \end{bmatrix}$$

So A indeed has the property that $AB = BA$ for *all* 2×2 matrices B.

Now we need to do the other direction. That is, we will now assume that the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has the property that $AB = BA$ for *all* 2×2 matrices B.
If this is the case, then if we let

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

we must have that $AB_1 = B_1A$ (if we assume that the equality holds for an arbitrary 2×2 matrix B then it certainly holds for any particular choice of a 2×2 matrix).

We can directly compute that

$$AB_1 = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix}$$

On the other hand, we can also compute that

$$B_1 A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}$$

But these two matrices must be equal by our hypothesis, so comparing the 1,2 entries of AB_1 and B_1A gives us $a_{12} = 0$, while a comparison of the 2,1 entries yields $a_{21} = 0$. Therefore we must have that

$$A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$$

By the same reasoning as above, if we let

$$B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

then we must have that $AB_2 = B_2A$. Directly computing gives us

$$AB_2 = \begin{bmatrix} a_{11} & a_{11} \\ a_{22} & a_{22} \end{bmatrix}$$

(Note that we have used the fact that $a_{12} = a_{21} = 0$, which we proved above). On the other hand

$$B_2 A = \begin{bmatrix} a_{11} & a_{22} \\ a_{11} & a_{22} \end{bmatrix}$$

Again, since these two matrices must be equal a comparison of either the 1,2 or the 2,1 entries of AB_2 and B_2A gives us that $a_{11} = a_{22}$, so indeed A must be of the form

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

This completes the proof.