HOMEWORK 2 SOLUTIONS

PROBLEM 1

(A) $\begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 10 \\ 7 & 8 & 17 \end{bmatrix}$

(C) $\begin{bmatrix} 14 & 16 \\ 8 & 9 \end{bmatrix}$ (D) $\begin{bmatrix} -16 & 9 \\ 0 & 22 \end{bmatrix}$

(E) $\begin{bmatrix} -2 & 28 \\ 12 & 8 \end{bmatrix}$ (F) $\begin{bmatrix} 0 & -1 & 1 \\ 12 & 5 & 17 \\ 19 & 0 & 22 \end{bmatrix}$

(G) NOT DEFINED (H) $\begin{bmatrix} 7 & 6 \\ 8 & 0 \\ 17 & 10 \end{bmatrix}$

PROBLEM 2

LET $A=(a_{ij})$ $B=(b_{ij})$

(A) $Tr[A^T] = \sum_{j=1}^{n} a_{jj}^T = \sum_{j=1}^{n} a_{jj} = Tr[A]$

(B) $Tr[A+B] = \sum_{j=1}^{n} (a_{jj}+b_{jj}) = \sum_{j=1}^{n} a_{jj} + \sum_{j=1}^{n} b_{jj} = Tr[A]+Tr[B]$

(C) Let $C=AB$ and $D=BA$.

Applying definition 1.7 (page 22 of book) we have

$$c_{jj} = \sum_{k=1}^{n} a_{jk}b_{kj}$$

So $Tr[AB] = \sum_{j=1}^{n} c_{jj} = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{jk}b_{kj}\right) = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} a_{jk}b_{kj}\right) = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} b_{kj}a_{jk}\right)$

$= \sum_{k=1}^{n} d_{kk} = Tr[BA]$

(D) Let $E = A^T A$.

Then again from definition 1.7 we have

$$e_{jj} = \sum_{k=1}^{n} a^T_{jk} a_{kj} = \sum_{k=1}^{n} a_{kj} a_{kj} = \sum_{k=1}^{n} (a_{kj})^2 \geq 0$$

(Clearly $a_{kj}^2 \geq 0$ for each k, j and the sum of positive terms is positive).

$$\text{Then } Tr[A^T A] = \sum_{j=1}^{n} e_{jj} = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{kj}^2\right) \geq 0$$

PROBLEM 3 Following the hint, we will compute

$$Tr[AB - BA] \quad \text{and} \quad Tr[I_n]$$

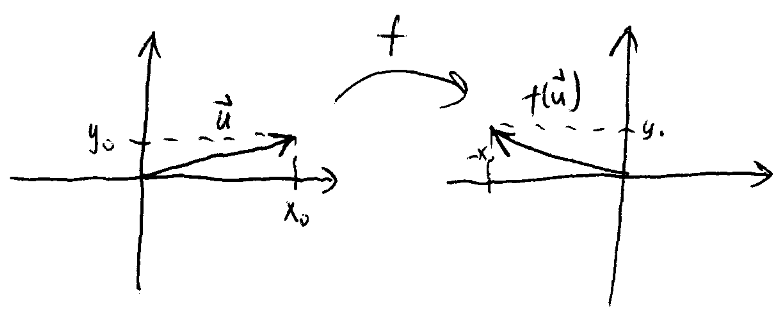
In 2(b) we showed the trace of a sum is the sum of the traces. An entirely analagous argument shows the trace of a difference is the difference of the traces. So $Tr[AB-BA] = Tr[AB] - Tr[BA]$.
But by Problem 2(c) we have $Tr[AB] = Tr[BA]$, so $Tr[AB-BA]$
$= Tr[AB] - Tr[BA] = 0$

On the other hand, direct computation shows $Tr[I_n] = \sum_{j=1}^{n} 1 = n \neq 0$

So $Tr[AB-BA] \neq Tr[I_n]$ and we conclude that $AB - BA \neq I_n$

PROBLEM 4

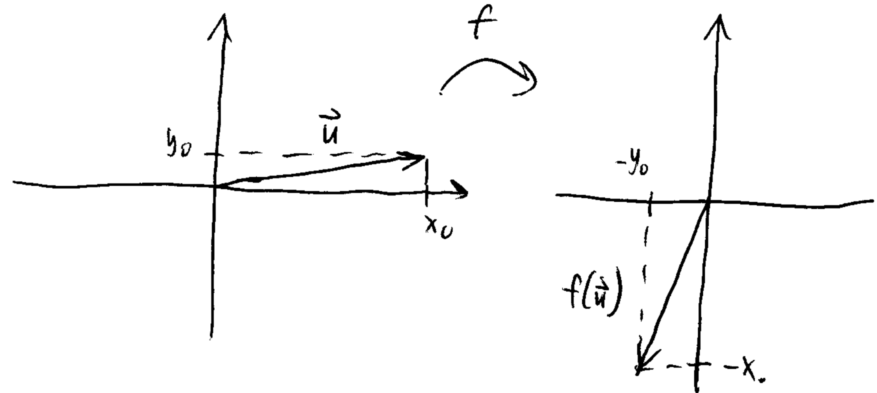
(A) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -x_0 \\ y_0 \end{bmatrix}$



REFLECTION ACROSS THE y-AXIS

HOMEWORK 2 SOLUTIONS (PAGE 3)

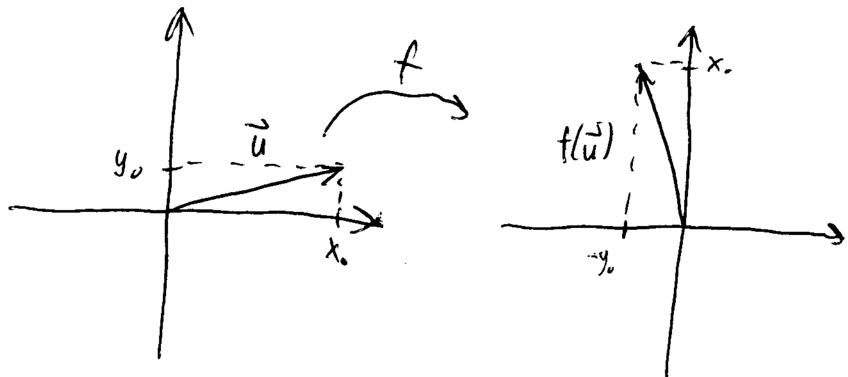
(B) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -y_0 \\ -x_0 \end{bmatrix}$



REFLECTION ACROSS THE LINE $y = -x$

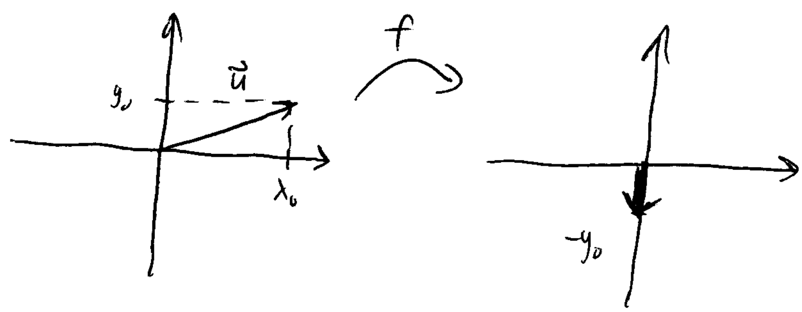
CAN ALSO DESCRIBE AS, e.g. ROTATION BY $\frac{3\pi}{2}$ FOLLOWED BY REFLECTION ACROSS THE y-AXIS.

(C) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -y_0 \\ x_0 \end{bmatrix}$



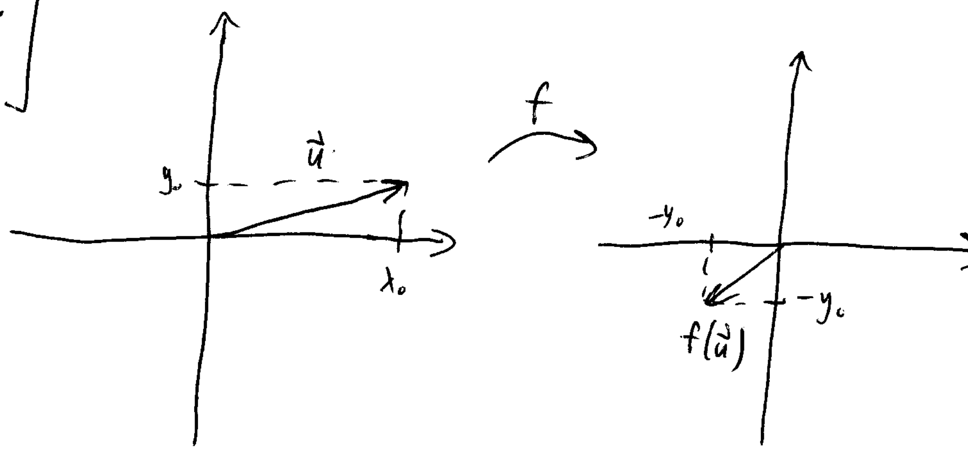
ROTATION (COUNTERCLOCKWISE) BY $\frac{\pi}{2}$ RADIANS

(D) $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ -y_0 \end{bmatrix}$



PROJECTION ONTO y-AXIS FOLLOWED BY REFLECTION ACROSS X-AXIS (CAN ALSO REFLECT, THEN PROJECT)

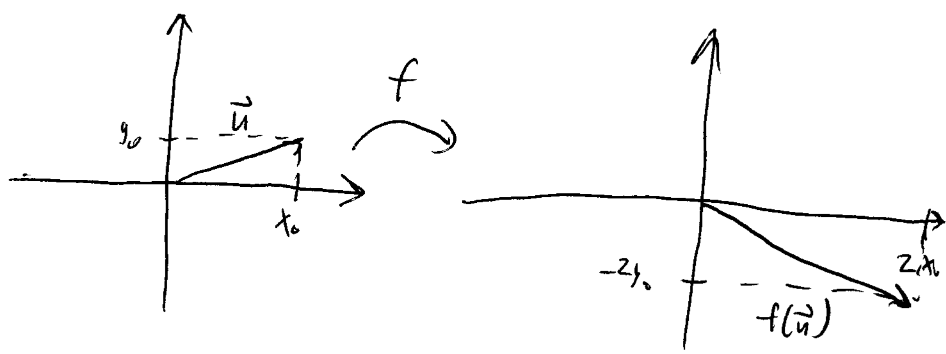
(E) $\begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -y_0 \\ -y_0 \end{bmatrix}$



BASIC DESCRIPTION: SENDS VECTORS TO LINE $x=y$

DETAILED DESCRIPTION: PROJECT ONTO y-AXIS, THEN ROTATE COUNTERCLOCKWISE BY $\frac{3\pi}{4}$, THEN DILATE BY A FACTOR OF $\sqrt{2}$

(F) $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 2x_0 \\ -2y_0 \end{bmatrix}$



REFLECTION ACROSS THE x-AXIS FOLLOWED BY DILATION BY A FACTOR OF 2 (OR DILATE THEN REFLECT)

PROBLEM 5

This follows from properties of matrix multiplication

$$f(\lambda\vec{u}+\mu\vec{v}) = A(\lambda\vec{u}+\mu\vec{v}) = A(\lambda\vec{u}) + A(\mu\vec{v}) = \lambda(A\vec{u}) + \mu(A\vec{v})$$

$= \lambda f(\vec{u}) + \mu f(\vec{v})$. We have used Theorem 1.2(c) (Page 35) and Theorem 1.3(d) (Page 37)

Homework 2 Solutions (Page 5)

Problem 6

(A)
$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

(B)
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(C)
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(D)
$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 7

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Depending on how you solve, you may have to treat the cases $\cos\theta \neq 0$ and $\cos\theta = 0$ separately

Problem 8

W IS A FREE VARIABLE

$x = -2 + r$
$y = -1$
$z = 8 - 2r$
$w = r$ (AN ARBITRARY NUMBER)

Problem 9

NO SOLUTION. THE SYSTEM IS INCONSISTENT

HOMEWORK 2 SOLUTIONS (PAGE 6)

PROBLEM 10 We must split into 2 cases.

Case 1: $A \neq 0$

By performing Gaussian elimination, we can convert the system into an equivalent system as follows

$$\left[\begin{array}{cc|c} A & B & 0 \\ C & D & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{B}{A} & 0 \\ C & D & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{B}{A} & 0 \\ 0 & D-\frac{BC}{A} & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & \frac{B}{A} & 0 \\ 0 & AD-BC & 0 \end{array}\right]$$

So we have the system $\begin{aligned} x + \tfrac{B}{A} y &= 0 \\ (AD-BC) y &= 0 \end{aligned}$ is equivalent to the original system

If $AD-BC=0$ then $x=\frac{B}{A}, y=-1$ is a (non-trivial) solution.

On the other hand if the system has a nontrivial solution then we either have a solution with $x \neq 0$ or a solution with $y \neq 0$ (or both).

If $y \neq 0$ then $(AD-BC)y=0 \Longrightarrow AD-BC=0$

If $x \neq 0$ then $x + \frac{B}{A} y = 0 \Longrightarrow y \neq 0$, which again gives us $AD-BC=0$ from the second equation.

Case 2: $A=0$

Then our system is
$$\begin{aligned} By &= 0 \\ Cx+Dy &= 0 \end{aligned}$$

and since $A=0$, $AD-BC=0$ is equivalent to $-BC=0$.

If $-BC=0$ then $B=0$ or $C=0$ (or both).

If $B=0$ then our system is
$$\begin{aligned} 0 &= 0 \\ Cx+Dy &= 0 \end{aligned}$$

which has a nontrivial solution (If $C\neq 0$ then $x=D, y=-C$ is a nontrivial solution. If $C=0$ then $x=1, y=0$ is a nontrivial solution.)

If $C=0$ then our system is
$$\begin{aligned} By &= 0 \\ Dy &= 0 \end{aligned}$$

which has the nontrivial solution $x=1, y=0$

On the other hand, if the system has a nontrivial solution then we have a solution with either $x\neq 0$ or $y\neq 0$ (or both).

If $y\neq 0$, then $By=0 \Rightarrow B=0 \Rightarrow -BC=0 \Rightarrow AD-BC=0$

If $x\neq 0$ then $Cx+Dy=0 \Rightarrow$ either $C=0$ or $y\neq 0$. If $C=0$ then $-BC=0$, so $AD-BC=0$ and if $y\neq 0$ we have already argued that $By=0$ then gives us $AD-BC=0$

PROBLEM 11 We can use Gaussian elimination to convert the system to the equivalent system
$$\begin{aligned} x-3y+2z &= c \\ y-\tfrac{1}{8}z &= \tfrac{1}{8}(b-3c) \\ 0 &= a-b+c \end{aligned}$$

This system is consistent if and only if $a-b+c=0$