HOMEWORK #3 SOLUTIONS

(1a) We can get $\{5,2,1,3,4\}$ from $\{1,2,3,4,5\}$ by swapping the elements in the following manner

$$\{1,2,3,4,5\} \rightarrow \{5,2,3,4,1\} \rightarrow \{5,2,1,4,3\} \rightarrow \{5,2,1,3,4\}$$

(Note: This is not the only way to achieve this)

We performed three swaps, so the permutation is <u>odd</u>

(b) $\{1,2,3,4,5\} \rightarrow \{4,2,3,1,5\} \rightarrow \{4,1,3,2,5\} \rightarrow \{4,1,3,5,2\}$

The permutation is <u>odd</u>

(c) $\{1,2,3,4,5\} \rightarrow \{5,2,3,4,1\} \rightarrow \{5,4,3,2,1\}$

The permutation is <u>even</u>

(d) $\{1,2,3,4,5,6\} \rightarrow \{6,2,3,4,5,1\} \rightarrow \{6,5,3,4,2,1\} \rightarrow \{6,5,4,3,2,1\}$

The permutation is ~~odd~~ even $\rightarrow \{6,5,4,1,2,3\}$

(e) Case 1: n even, so $n=2k$ for some integer k

$\{1,2,\dots,k-1,k,k+1,k+2,\dots,2k-1,2k\} \rightarrow$ ~~$\{2k,2,\dots,k-1,k,k+1,k+2\} \rightarrow \{2k,2k-1,\dots,k+1,k,$~~

$\{2k,2,\dots,k-1,k,k+1,k+2,\dots,2k-1,1\} \rightarrow \{2k,2k-1,\dots,k-1,k,k+1,k+2,\dots,2,1\}$

$\rightarrow \dots \rightarrow \{2k,2k-1,\dots,k+2,k,k+1,k-1,\dots,2,1\}$

$\rightarrow \{2k,2k-1,\dots,k+2,k+1,k,k-1,\dots,2,1\}$. We performed k swaps so the permutation is even ~~if~~ k is even (i.e. $k=2\ell$ for some integer ℓ so that $n=4\ell$) and odd if k is odd (i.e. $k=2\ell+1$ so that $n=4\ell+2$)

Case 2: n odd, so $n=2k+1$. A similar argument as above shows we'll make k swaps, so the permutation is even if $k=2\ell$ (so $n=4\ell+1$) and odd if $k=2\ell$ (so $n=4\ell+3$)

Conclusion: The permutation $\{1,2,\dots,n-1,n\} \rightarrow \{n,n-1,\dots,2,1\}$ is <u>even</u> if $n=4\ell$ or $n=4\ell+2$ and <u>odd</u> if $n=4\ell+1$ or $n=4\ell+3$ (for some integer ℓ).

(2a)

$$\begin{array}{ccccc} 2 & 1 & 3 & 2 & 1 \\ 3 & 2 & 1 & 3 & 2 \\ 0 & 1 & 2 & 0 & 1 \end{array}$$

$$-0 - 2 - 6 + 8 + 0 + 9 = 9$$

So $\det \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \boxed{9}$

We can use this result to quickly obtain the answers to parts (b) and (c)

(b) $\det \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \boxed{9}$

because $\begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ is the transpose of $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

(c) $\det \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} = -\det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix} = (-1)^2 \det \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$= (-1)^2 (9) = \boxed{9}$

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(d)

$$\det\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix} = -\det\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \det\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$= (6)(2)(4)(3) = \boxed{144}$$

(e)

$$\det\begin{bmatrix} 0 & 0 & 4 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \end{bmatrix} = -\det\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \end{bmatrix} = \det\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$= -\det\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = -(6)(2)(4)(3) = \boxed{-144}$$

(f)

$$\det\begin{bmatrix} 0 & 0 & 0 & 4 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix} = -\det\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix} = \det\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 6 & 0 \end{bmatrix}$$

$$= -\det\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = -(3)(2)(6)(4) = \boxed{-144}$$

(3A)

$$\det\begin{bmatrix} \lambda-1 & 2 \\ 3 & \lambda-2 \end{bmatrix} = (\lambda-1)(\lambda-2)-6 = \lambda^2-3\lambda+4 = (\lambda-4)(\lambda+1)$$

$$\boxed{\lambda_1 = 4,\ \lambda_2 = -1}$$

(B)

$$\det\begin{bmatrix} \lambda-1 & -1 & -2 \\ 0 & \lambda & 2 \\ 0 & 0 & \lambda-3 \end{bmatrix} = (\lambda-1)\lambda(\lambda-3)$$

$$\boxed{\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 3}$$

(C)

$$\det\begin{bmatrix} \lambda-1 & 0 & 1 \\ -2 & \lambda & -1 \\ 0 & 0 & \lambda+1 \end{bmatrix} = (\lambda-1)\lambda(\lambda+1)$$

$$\boxed{\lambda_1 = -1,\ \lambda_2 = 0,\ \lambda_3 = 1}$$

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(4) $AB = I_n \Rightarrow \det(AB) = \det(I_n) = 1$

$\Rightarrow \det(A)\det(B) = 1 \Rightarrow \det(A) \neq 0, \det(B) \neq 0$

(5) $A = A^{-1} \Rightarrow AA = I_n \Rightarrow \det(AA) = \det(I_n) = 1$

$\Rightarrow [\det(A)]^2 = 1 \Rightarrow \det(A) = \pm 1$

Examples:

$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $A_1^{-1} = A_1$ and $\det(A_1) = 1$

$A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. $A_2^{-1} = A_2$ and $\det(A_2) = -1$

(6) $A^2 = A \Rightarrow \det(A^2) = \det(A) \Rightarrow \det(AA) = \det(A)$

$\Rightarrow \det(A)\det(A) = \det(A)$

A is invertible so $\det(A) \neq 0$. Divide both sides by $\det(A)$.

$$\det(A) = 1$$

Alternate argument: Multiply both sides of $A^2 = A$ by A^{-1}.

$A^2A^{-1} = AA^{-1} = I_n$. But $A^2A^{-1} = A(AA^{-1}) = AI_n = A$. So $A = I_n$ and so $\det(A) = 1$

(7A) We have

$$M_{1,2} = \begin{bmatrix} 1 & -2 & 4 \\ -1 & -3 & -2 \\ 0 & -1 & 5 \end{bmatrix}$$

So $\det[M_{1,2}]$ is given by

$$\begin{array}{ccccc} 1 & -2 & 4 & 1 & -2 \\ -1 & -3 & -2 & -1 & -3 \\ 0 & -1 & 5 & 0 & -1 \end{array}$$

$$-0 - 2 - 10 + (-15) + 0 + 4 = -23$$

$$\det[M_{1,2}] = -23$$

We can similarly compute $\det[M_{3,3}] = 15$ and $\det[M_{4,1}] = -28$

(b) $A_{2,3} = (-1)^{2+3}\det[M_{2,3}] = (-1)^5(7) = -7$

$A_{1,4} = (-1)^{1+4}\det[M_{1,4}] = (-1)^5(7) = -7$

$A_{3,1} = (-1)^{3+1}\det[M_{3,1}] = (-1)^4(12) = 12$

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(8A) $\det(\lambda I_n - A)$ is a sum of products of polynomials and hence itself a polynomial. It has degree n since the most λ's show up in the term $(\lambda - a_{11})(\lambda - a_{22})\cdots(\lambda - a_{nn})$

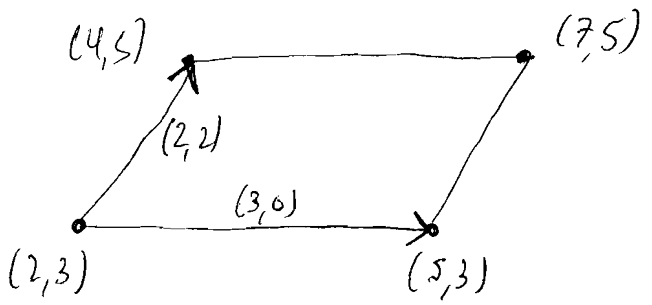
$$= \lambda^n + \{\,\}\lambda^{n-1} + \{\,\}\lambda^{n-2} + \dots + \{\,\}\lambda + \{\,\}$$

(B) 1, as we can see above

(C) $\det(-A)$

(D) $\mathrm{Tr}(-A)$ $\quad$ ($-\mathrm{Tr}(A)$ also works)

(9)



$$\text{Area}(\text{▱}) = \left|\det\begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}\right| = 6$$

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10A

$$A^{-1} = \begin{bmatrix} -\frac{1}{32} & \frac{3}{32} & \frac{9}{32} \\ \frac{10}{32} & \frac{2}{32} & \frac{6}{32} \\ \frac{2}{32} & -\frac{6}{32} & \frac{14}{32} \end{bmatrix}$$

B

$$B^{-1} = \begin{bmatrix} \frac{24}{150} & -\frac{42}{150} & -\frac{30}{150} \\ \frac{19}{150} & -\frac{2}{150} & -\frac{30}{150} \\ -\frac{4}{150} & \frac{32}{150} & \frac{30}{150} \end{bmatrix}$$

C

$$C^{-1} = \begin{bmatrix} \frac{30}{28} & \frac{5}{28} & -\frac{9}{28} & -\frac{46}{28} \\ \frac{32}{28} & -\frac{4}{28} & -\frac{4}{28} & -\frac{36}{28} \\ \frac{12}{28} & \frac{2}{28} & \frac{2}{28} & -\frac{24}{28} \\ -\frac{16}{28} & \frac{2}{28} & \frac{2}{28} & \frac{32}{28} \end{bmatrix}$$