

# Math 340 Solutions to Assignment 5

Marc Conrad

December 13, 2008

## Problem 1.

(a) Yes, as the following computations show.

$$\begin{aligned}L[(x_1 + x_2, y_1 + y_2, z_1 + z_2)] &= (3[x_1 + x_2] - [y_1 + y_2], z_1 + z_2, [x_1 + x_2] + [y_1 + y_2]) \\ &= (3x_1 - y_1, z_1, x_1 + y_1) + (3x_2 - y_2, z_2, x_2 + y_2) \\ &= L(x_1, y_1, z_1) + L(x_2, y_2, z_2)\end{aligned}$$

$$\begin{aligned}L[\lambda(x, y, z)] &= (3\lambda x - \lambda y, \lambda z, \lambda x + \lambda y) \\ &= \lambda(3x - y, z, x + y) \\ &= \lambda L(x, y, z)\end{aligned}$$

(b) No. Consider the two vectors  $(1, 0, 0)$  and  $(1, 1, 0)$ . Then we have

$$\begin{aligned}L[(1, 0, 0) + (1, 1, 0)] &= L(2, 1, 0) = (4, 1, 0) \\ L(1, 0, 0) + L(1, 1, 0) &= (1, 0, 0) + (1, 1, 0) = (2, 1, 0) \neq (4, 1, 0)\end{aligned}$$

We can apply similar reasoning to solve parts (c) through (h). The answers are

(c) No.

(d) Yes. Note that the effect of this linear transformation is to take the derivative of the polynomial.

(e) Yes. Note that the effect of this linear transformation is to take the antiderivative of the polynomial.

(f) Yes.

(g) No.

(h) Yes.

**Problem 2.** Let  $\mathbf{x}_1$  and  $\mathbf{x}_2 \in R^n$ .

Then by the distributive property of matrix multiplication  $L[\mathbf{x}_1 + \mathbf{x}_2] = A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = L[\mathbf{x}_1] + L[\mathbf{x}_2]$ .

Now let  $\mathbf{x} \in R^n$  and  $\lambda \in R$ .

Since we can move scalars around at will when multiplying them by matrices, we have  $L[\lambda\mathbf{x}] = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda L[\mathbf{x}]$ .

So  $L$  is indeed a linear transformation.

**Problem 3.**

(a) Let  $f, g \in V$ . Then

$$\begin{aligned} L[f + g](x) &= a_0(x)(f + g)''(x) + a_1(x)(f + g)'(x) + a_2(x)(f + g)(x) \\ &= a_0(x)f''(x) + a_1(x)f'(x) + a_2(x)f(x) + a_0(x)g''(x) + a_1(x)g'(x) + a_2(x)g(x) \\ &= L[f](x) + L[g](x) \end{aligned}$$

Now let  $f \in V$  and  $\lambda \in R$ . Then

$$\begin{aligned} L[\lambda f](x) &= a_0(x)(\lambda f)''(x) + a_1(x)(\lambda f)'(x) + a_2(x)(\lambda f)(x) \\ &= \lambda[a_0(x)f''(x) + a_1(x)f'(x) + a_2(x)f(x)] \\ &= \lambda L[f](x) \end{aligned}$$

So indeed  $L$  is a linear transformation.

(b) No. For example, let  $f(x) = x$  and  $\lambda = 2$ . Then

$$\begin{aligned} L[\lambda f](x) &= a_0(x)(2)^2 + a_1(x)2 + a_2(x)(2x) \\ &= 4a_0(x) + 2a_1(x) + 2xa_2(x) \end{aligned}$$

While on the other hand

$$\lambda L[f](x) = 2a_0(x) + 2a_1(x) + 2xa_2(x) \neq L[\lambda f](x)$$

(c) We can generalize the result of part (a) to conclude that an  $n$ -th order differential operator is linear if it is of the form

$$L[f](x) = a_0(x)f^{(n)}(x) + a_1(x)f^{(n-1)}(x) + \dots + a_{n-1}(x)f'(x) + a_n(x)f(x)$$

**Problem 4.**

Let  $\mathbf{v} \in V$ . Then there is a unique way to write  $\mathbf{v}$  in terms of the basis consisting of the  $\mathbf{e}_j$ , say

$$\mathbf{v} = a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n$$

Now define

$$L(\mathbf{v}) = a_1\mathbf{f}_1 + \dots + a_n\mathbf{f}_n$$

It is clear from the definition of  $L$  that  $L(\mathbf{e}_j) = \mathbf{f}_j$ . It remains to show (1) that  $L$  is a linear transformation and (2) it's the only linear transformation such that  $L(\mathbf{e}_j) = \mathbf{f}_j$ .

(1) Let  $\mathbf{v}, \mathbf{w} \in V$ . Then there are scalars  $a_j$  and  $b_j$  so that

$$\begin{aligned}\mathbf{v} &= a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n \\ \mathbf{w} &= b_1\mathbf{e}_1 + \dots + b_n\mathbf{e}_n\end{aligned}$$

Then

$$\begin{aligned}L(\mathbf{v} + \mathbf{w}) &= (a_1 + b_1)\mathbf{f}_1 + \dots + (a_n + b_n)\mathbf{f}_n \\ &= a_1\mathbf{f}_1 + \dots + a_n\mathbf{f}_n + b_1\mathbf{f}_1 + \dots + b_n\mathbf{f}_n \\ &= L(\mathbf{v}) + L(\mathbf{w})\end{aligned}$$

Also, if  $\lambda \in R$  we can similarly show that

$$L(\lambda \mathbf{v}) = \lambda L(\mathbf{v})$$

So  $L$  is a linear transformation with the desired properties.

(2) To show uniqueness, suppose  $T$  is also a linear transformation with the desired properties. Then if we apply  $T$  to  $\mathbf{v}$  from above, by the properties of linear transformations we have that

$$\begin{aligned} T(\mathbf{v}) &= L(a_1 \mathbf{e}_1 + \dots + a_n \mathbf{e}_n) \\ &= a_1 L(\mathbf{e}_1) + \dots + a_n L(\mathbf{e}_n) \\ &= a_1 \mathbf{f}_1 + \dots + a_n \mathbf{f}_n \\ &= L(\mathbf{v}) \end{aligned}$$

Therefore  $T = L$  and so  $L$  is the **unique** linear transformation with the stated properties.

**Problem 5.**

(a)  $L[(2, 3, -2, 3)] = (0, 6) \neq \mathbf{0}$

So  $(2, 3, -2, 3)$  is not in the kernel of  $L$ .

(b)  $L[(4, -2, -4, 2)] = (0, 0)$

So  $(4, -2, -4, 2)$  is in the kernel of  $L$ .

(c) Yes. For example,  $L(1, 2, 0, 0) = (1, 2)$

(d) Yes. For example  $L(0, 0, 0, 0) = (0, 0)$ .

In fact, one can show that for any linear transformation  $T$ ,  $T(\mathbf{0}) = \mathbf{0}$ , so  $\mathbf{0}$  is always an element of the range of a given linear transformation.

(e)  $\{(u_1, u_2, u_3, u_4) \in R^4 \mid u_1 = -u_3, u_2 = -u_4\}$

(f)  $R^2$

**Problem 6**

(a) The elements in the kernel are the constant polynomials  $P(t) = a_0$  (this is because the derivative of a constant is zero). From this we see that elements in the kernel can be expressed as a linear combination of the polynomial  $P(t) = 1$  and therefore a basis for the kernel is  $\{1\}$  and its dimension is 1.

(b) If  $P(t) = a_0 + a_1t + a_2t^2$ , then  $L[P](t) = t^2P'(t) = a_1t^2 + 2a_2t^3$ . From this we see that elements in the range can be expressed as linear combinations of the polynomials  $t^2$  and  $t^3$ . Furthermore, these two polynomials form a linearly independent set, so a basis for the kernel is  $\{t^2, t^3\}$  and its dimension is 2.

**Problem 7.**

(a) Suppose that

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$
$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Then we have that

$$L[\mathbf{x}] = A\mathbf{x}$$
$$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

From this we see that the range is the set of all linear combinations of the columns of  $A$ , which is exactly the definition of the column space of  $A$ .

(b) Apply part (a) to  $A^T$  and use the fact that the rows of  $A$  are the columns of  $A^T$ .

**Problem 8.**

We find the kernel by solving the system of equations represented by the matrix

$$\left[ \begin{array}{ccccc|c} 1 & 0 & -1 & 3 & 1 & 0 \\ 1 & 0 & 0 & 2 & -1 & 0 \\ 2 & 0 & -1 & 5 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array} \right]$$

In row echelon form this becomes

$$\left[ \begin{array}{ccccc|c} 1 & 0 & -1 & 3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

The most general solution is then given by  $x_1 = -2x_4, x_2 = x_2, x_3 = x_4, x_4 = x_4, x_5 = 0$ . We can write this as

$$\mathbf{x} = x_2(0, 1, 0, 0, 0) + x_4(-2, 0, 1, 1, 0)$$

So the two vectors  $(0, 1, 0, 0, 0)$  and  $(-2, 0, 1, 1, 0)$  span the kernel. They are also linearly independent, so a basis for the kernel is  $\{(0, 1, 0, 0, 0), (-2, 0, 1, 1, 0)\}$  and it has dimension 2.

(b) By Problem 7(a), the columns of the given matrix span the range. However, they are linearly dependent. We can extract a basis by successively removing vectors that can be expressed as linear combinations of the others. To do this, write the zero vector as a linear combination of the columns, yielding a system of equations represented by the augmented matrix

$$\left[ \begin{array}{ccccc|c} 1 & 0 & -1 & 3 & 1 & 0 \\ 1 & 0 & 0 & 2 & -1 & 0 \\ 2 & 0 & -1 & 5 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array} \right]$$

In row echelon form this becomes

$$\left[ \begin{array}{ccccc|c} 1 & 0 & -1 & 3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

One can remove each vector whose column does not contain a leading 1 (in this case the second and fourth columns) and the result will be a basis. So a basis for the range is  $\{(1, 1, 2, 0), (-1, 0, -1, -1), (-1, -1, -1, 0)\}$  and the dimension is 3.

**Problem 9.**

(a) Suppose we have written the zero vector as a linear combination of the  $\mathbf{v}$ . Then

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

Since  $L$  sends zero to zero we must then have that

$$L[\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n] = \mathbf{0}$$

By the properties of linear transformations, this gives us

$$\alpha_1 L[\mathbf{v}_1] + \dots + \alpha_n L[\mathbf{v}_n] = \mathbf{0}$$

But since  $\{L[\mathbf{v}_1], \dots, L[\mathbf{v}_n]\}$  are linearly independent, this implies that  $\alpha_1 = \dots = \alpha_n = 0$ , which implies that  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  are linearly independent.

(b) No. For example, consider the transformation  $L : R^2 \rightarrow R^2$  given by  $L[(x, y)] = (x + y, x + y)$ . Then  $\{(1, 0), (0, 1)\}$  is a linearly independent set, but  $L[(1, 0)] = (1, 1)$  and  $L[(0, 1)] = (1, 1)$  so  $\{L[(1, 0)], L[(0, 1)]\}$  is a linearly dependent set.

**Problem 10.**

The linear transformation  $L[(x, y)] = (x + 3y, -x + y, 2x - y)$  has the desired properties.

**Problem 11.**

$$(a) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$(c) \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$(d) \begin{bmatrix} 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 3 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

**Problem 12.**

$$(a) \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$(b) \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}$$

$$(c) \begin{bmatrix} 3 & 2 & 3 \\ 0 & 2 & 2 \end{bmatrix}$$

$$(d) \begin{bmatrix} \frac{3}{2} & 2 & \frac{5}{2} \\ \frac{3}{2} & 0 & \frac{1}{2} \end{bmatrix}$$