# Solutions to Math 222 Review Problems for Exam 1

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**Question 1.**

(a) $x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C$. Use integration by parts. The technique of tabular integration will save you some time.

(b) $-\dfrac{1}{4(x^2+1)^2} + C$. The easiest way to do the problem is the substitution $u = x^2 + 1$. You can also use the trigonometric substitution $x = \tan\theta$. The method of partial fractions, however, will not help on this problem because the integrand is already in simplified form (i.e. a partial fraction decomposition will just give us the same integral that we started with).

(c) $\dfrac{1}{2}\tan^2 x + \ln|\cos x| + C$. Rewrite the integrand as $(\sec^2 x - 1)\tan x$ or $\sec^2 x \tan x - \tan x$. The first part can be integrated using the substitution $u = \tan x$ and the second part can be solved by rewriting as $\frac{\sin x}{\cos x}$ and using the substitution $u = \cos x$.

(d) $x\tan^{-1} x - \dfrac{1}{2}\ln(1+x^2) + C$. Use integration by parts with $u = \tan^{-1} x$ and $dv = dx$.

(e) $\dfrac{1}{9}\ln|x-1| + \dfrac{1}{3(x-1)} + \dfrac{17}{9}\ln|x+2| + C$. Use the method of partial fractions. The correct decomposition is of the form $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2}$.

(f) $\dfrac{1}{3}x^3 \ln x - \dfrac{1}{9}x^3 + C$. Use integration by parts with $u = \ln x$ and $dv = x^2\,dx$.

(g) $\dfrac{1}{2}x\sqrt{x^2-1} + \dfrac{1}{2}\ln|x + \sqrt{x^2-1}| + C$. The trigonometric substitution $x = \sec\theta$ will transform the integrand into $\sec^3\theta$. Then use integration by parts with $u = \sec\theta$ and $dv = \sec^2\theta\,d\theta$ (see example 6 on page 584 of the book). After integrating, use a reference triangle to get the answer in terms of $x$.

(h) $\dfrac{1}{5}e^{2x}\sin x + \dfrac{2}{5}e^{2x}\cos x + C$. Using integration by parts twice will give you $\displaystyle\int e^{2x}\cos x\,dx = e^{2x}\sin x + 2e^{2x}\cos x - 4\int e^{2x}\cos x\,dx$. One can solve this equation by adding the last term to both sides.

(i) $\dfrac{4}{3}\sin^3 x - \dfrac{4}{5}\sin^5 x + C$. Use the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ to rewrite the integrand as $4\sin^2 x\cos^3 x$. This is of the form $\sin^m x\cos^n x$ with $m$ even and $n$ odd, so rewrite the integrand as $4\sin^2 x(1-\sin^2 x)\cos x$ and then use the substitution $u = \sin x$.

(j) $\dfrac{1}{2}x^2 + \ln|x| - \ln(x^2+1) + C$. This is an improper fraction (the degree of the numerator is greater than or equal to the degree of the denominator) so first we must use polynomial long division to rewrite our integrand as $x + \dfrac{-x^2+1}{x^3+x}$. The integral of the first part is just $\frac{1}{2}x^2$. Noting that the denominator of the remaining rational function factors as $x(x^2+1)$ we can integrate that part using partial fractions. The correct form of the decomposition is $\dfrac{-x^2+1}{x^3+x} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$.

(k) $4\sin^{-1}\left(\dfrac{x}{2}\right) + x\sqrt{4-x^2} + C$. Using the trigonometic substitution $x = 2\sin\theta$ we get that the answer (in terms of $\theta$) is $4\theta + 2\sin(2\theta)$. To get back in terms of $x$, first apply the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ and then use a reference triangle.

**Question 2.**

(a) $2\sqrt{2}\pi + \ln(\sqrt{2}+1) - \ln(\sqrt{2}-1)$. Applying the formula for surface area from Page 438 of the book gives that the surface area is

$$S = \int_0^\pi 2\pi \sin x \sqrt{1+\cos^2 x}\, dx$$

To solve this integral, first make the substitution $u = \cos x$ then make the trigonometic substitution $u = \tan\theta$.

(b) $\frac{3}{8}\pi^2$. Applying the formula for volume given on page 399 of the book gives

$$V = \int_0^\pi \pi \sin^4 x\, dx$$

This is of the form $\sin^m x \cos^n x$ with $n$ and $m$ both even. To solve, write $\sin^4 x = (\sin^2 x)^2 = (\frac{1}{2} - \frac{1}{2}\cos(2x))^2$. Simplify the resulting expression by using $\cos^2(2x) = \frac{1}{2} + \frac{1}{2}\cos(4x)$. Integrate the resulting function.

(c) $(\frac{\pi}{2}, \frac{3}{8})$. The $x$-coordinate comes from considering the symmetries of $y = \sin^2 x$. To get the $y$-coordinate, consider thin vertical strips as in Section 6.4 to get that the $y$-coordinate of the center of gravity is

$$\frac{\int_0^\pi \frac{1}{2}\sin^4 x\, dx}{\int_0^\pi \sin^2 x\, dx}$$

The integral in the numerator is very similar to the one in part (b), and the way to find it is essentially the same. Similar techniques will give us the value of the integral in the denominator.

**Question 3.** Note that this problem gives us a numerical estimate for $\ln 4$, since the exact value of the integral is $2\ln 2$ or $\ln 4$.

(a) Our estimate is $\dfrac{1171}{840}$ or about 1.394. Our bound for the error is $\dfrac{1}{48}$, or about 0.021.

(b) Our estimate is $\dfrac{1747}{1260}$ or about 1.387. Our bound for the error is $\dfrac{1}{960}$, or about 0.001.

**Question 4.**

(a) 1. You will need to use the fact that $\lim_{b\to\infty} be^{-b} = 0$. You can establish the limit using l'Hospital's rule.

(b) $2\sqrt{e-1}$. Use the substitution $u = e^x - 1$. Note that this is an improper integral because $\displaystyle\lim_{x\to 0^+} \frac{e^x}{\sqrt{e^x - 1}} = \infty$.

**Question 5.**

(a) Converges. Use the Direct Comparison Test with $\frac{1}{x^2}$.

(b) Converges. Establish this by computing the integral explicitly using the substitution $u = \ln x$. The integral happens to be equal to $\frac{3}{2}(\ln 2)^{2/3}$.

(c) Diverges. Use the Direct Comparison Test with $\frac{1}{x}$.

**Question 6.** The foci are at $(\pm\sqrt{5}, 0)$. The vertices are at $(\pm 3, 0)$. The directices are at $x = \pm\frac{9}{\sqrt{5}}$ and the eccentricity is $\frac{\sqrt{5}}{3}$. This is an ellipse centered at the origin passing through $(\pm 3, 0)$ and $(0, \pm 2)$.

**Question 7.** The eccentricity is $\sqrt{2}$. The vertices are at $(\pm\sqrt{2}, 0)$. The asymptotes are $x = \pm y$. This is a hyperbola given by the equation $\dfrac{x^2}{2} - \dfrac{y^2}{2} = 1$.

**Question 8.** The given equation is for a parabola. The eccentricity is 1 because all parabolas have eccentricity 1. The vertex is at the origin. The focus is at $(0, 1)$ and the directrix is $y = -1$.