Math 222  Quiz 6  November 4, 2009

Name:

Solve the following problems. Circle your final answer. You must show your work to earn full credit. Please make sure that your work is clear and legible. All work on the page will be assessed unless it is crossed out.

Question 1 (5 points). Find the sum of the following series.

\[ \sum_{n=0}^{\infty} \left( \frac{3}{5^n} + \frac{(-1)^n}{3^n} \right) \]

Solution: We have that \( \sum_{n=0}^{\infty} \frac{3}{5^n} = \sum_{n=0}^{\infty} 3 \left( \frac{1}{5} \right)^n \) is a convergent geometric series with \( a = 3 \) and \( r = \frac{1}{5} \) and so its sum is \( \frac{3}{1 - (1/5)} = \frac{15}{4} \).

We also have that \( \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} = \sum_{n=0}^{\infty} \left( \frac{-1}{3} \right)^n \) is a convergent geometric series with \( a = 1 \) and \( r = -\frac{1}{3} \) and so its sum is \( \frac{1}{1 + (1/3)} = \frac{3}{4} \).

Then, applying the sum rule for convergent series gives us

\[ \sum_{n=0}^{\infty} \left( \frac{3}{5^n} + \frac{(-1)^n}{3^n} \right) = \frac{15}{4} + \frac{3}{4} = \frac{18}{4} = \frac{9}{2} \]
Question 2 (5 points). Determine whether the following series converges or diverges.

\[ \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/3}} \]

Solution: We will use the integral test. Note that the function \( f(x) = \frac{1}{x(\ln x)^{1/3}} \) is continuous, positive and decreasing on \([2, \infty)\). Furthermore, \( f(n) = \frac{1}{n(\ln n)^{1/3}} \).

We now must compute \( \int_{2}^{\infty} f(x)\,dx \). Note that by making the substitution \( u = \ln x \) we get \( du = \frac{dx}{x} \) and so

\[
\int f(x)\,dx = \int \frac{dx}{x(\ln x)^{1/3}} \\
= \int u^{-1/3}\,du \\
= \frac{3}{2}u^{2/3} + C \\
= \frac{3}{2}(\ln x)^{2/3} + C
\]

Therefore we have that

\[
\int_{2}^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_{2}^{b} \frac{dx}{x(\ln x)^{1/3}} \\
= \lim_{b \to \infty} \left[ \frac{3}{2}(\ln x)^{2/3} \right]_{2}^{b} \\
= \lim_{b \to \infty} \frac{3}{2}(\ln b)^{2/3} - \frac{3}{2}(\ln 2)^{2/3} \\
= \infty
\]

Therefore \( \int_{2}^{\infty} f(x)\,dx \) diverges. Then, by the integral test, we can conclude that \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/3}} \) also diverges.