Notes on Integration

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1 Introduction

There are a number of errors that are often made by calculus students while they are learning integration. The purpose of this article is to discuss some of these errors so that you will know how to avoid them in advance.

2 Common Error # 1: Ignoring Composite Functions

Composite functions often create difficulties when integrating. While we can differentiate compositions of differentiable functions by applying the Chain Rule, there is no set rule for integrating compositions of functions; we must deal with them on a case-by-case basis. A common error is to simply ignore that there is a composition of functions and simply integrate the "outside" function. As an example, consider the problem of integrating $\sin^2(x)$.

Incorrect:

$$\int \sin^2(x)\,dx = \frac{1}{3}\sin^3(x) + C$$

The problem is that we cannot simply treat $\sin(x)$ as though it were a varaible. We need to find some alternate method of solving the problem. In this case, a trigonometric identity does the trick.

Correct:

$$\int \sin^2(x)\,dx = \int \frac{1-\cos(2x)}{2}\,dx = \frac{1}{2}\int (1-\cos(2x))\,dx =$$

$$\frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right) + C = \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

3 Common Error # 2: Incomplete u-substitution

In order for a u-substitution to work, we must make our substitution in such a way that the function we're integrating is expressed entirely in terms of the variable u. As an example, consider the problem of integrating $x^5(x^3+1)^{1/3}$.

Incorrect:

$$\int x^5(x^3+1)^{1/3}\,dx$$

$u = x^3+1$, and therefore $du = 3x^2\,dx$, so $\frac{1}{3}\,du = x^2\,dx$

$$\int x^5(x^3+1)^{1/3}\,dx = \int x^3\cdot x^2(x^3+1)^{1/3}\,dx = \int x^3\cdot\frac{1}{3}u^{1/3}\,du$$

$$= \frac{1}{3}x^3\int u^{1/3}\,du = \frac{1}{3}x^3\cdot\frac{3}{4}u^{4/3}+C = \frac{1}{4}x^3(x^3+1)^{4/3}+C$$

The problem here is that we are treating x as though it were a constant, when in fact it depends on u. The key to this problem is recognizing that the relationship between x and u allows us to eliminate that pesky x^3. That is, since $u = x^3+1$, we have that $x^3 = u-1$. Therefore we can substitute in $u-1$ where we have the stray term of x^3.

Correct:

$$\int x^5(x^3+1)^{1/3}\,dx$$

$u = x^3+1$, and therefore $du = 3x^2\,dx$, so $\frac{1}{3}\,du = x^2\,dx$, and $x^3 = u-1$

$$\int x^5(x^3+1)^{1/3}\,dx = \int x^3\cdot x^2(x^3+1)^{1/3}\,dx = \int (u-1)u^{1/3}\cdot\frac{1}{3}\,du =$$

$$\frac{1}{3}\int u^{4/3}-u^{1/3}\,du = \frac{1}{3}\left[\frac{3}{7}u^{7/3}-\frac{3}{4}u^{4/3}\right]+C = \frac{1}{7}(x^3+1)^{7/3}-\frac{1}{4}(x^3+1)^{4/3}+C$$

4 Common Error # 3: Improper u-substitution

When making a u-substitution, we must be very careful to make sure that what we end up with for du is something that is multiplied by the entire function. A common mistake is to choose something that's inside a radical, or a single term of a sum, as a du. For example, consider the problem of integrating $\sqrt{\sin^2 x \cos x}$ from 0 to $\frac{\pi}{2}$

Incorrect:

$$\int_0^{\pi/2} \sqrt{\sin^2 x \cos x}\, dx$$

Let $u = \sin x$, so $du = \cos x\, dx$. When $x = 0$, $u = 0$. When $x = \frac{\pi}{2}$, $u = 1$.

$$\int_0^{\pi/2} \sqrt{\sin^2 x \cos x}\, dx = \int_0^1 \sqrt{u^2}\, du = \int_0^1 u\, du = \left[\frac{1}{2}u^2\right]_0^1 = \frac{1}{2}$$

The problem here is that there isn't actually a $\cos x\, dx$ in the integrand. What is there is $\sqrt{\cos x}\, dx$, which is not the same thing. In this case, we can rewrite the integral by noticing that $\sqrt{\sin^2 x} = |\sin x|$, and since $\sin x \geq 0$ on $[0, \frac{\pi}{2}]$, we can replace $\sqrt{\sin^2 x}$ with $\sin x$. This will allow us to solve the integral.

Correct:

$$\int_0^{\pi/2} \sqrt{\sin^2 x \cos x}\, dx = \int_0^{\pi/2} \sin x \sqrt{\cos x}\, dx$$

Let $u = \cos x$. Then $du = -\sin x\, dx$, so $-du = \sin x\, dx$. When $x = 0$, $u = 1$. When $x = \frac{\pi}{2}$, $u = 0$.

$$\int_0^{\pi/2} \sin x \sqrt{\cos x}\, dx = -\int_1^0 \sqrt{u}\, du = -\left[\frac{2}{3}u^{3/2}\right]_1^0 = \frac{2}{3}$$

5 Common Error # 4: Incorrect Method

When dealing with composite functions, there is a tendency to want to always use u-substitution to solve the problem. However, there are cases where the method simply does not apply. Consider the example of trying to solve $\int_{-1}^1 \sqrt{1 - x^2}\, dx$. Trying the substitution $u = 1 - x^2$ to get rid of the term underneath the radical may seem like the natural thing to do, but it won't work because then we would have $du = 2x\, dx$, and there's no x term outside the radical. In this case we must try a different method. This integral turns out to have a very simple geometric interpretation. It is the area of a semicirlce of radius 1. Therefore we can conclude that the integral is equal to $\frac{\pi}{2}$ without doing any calculations.