NAME:

Complete each of the following problems. Show all work. You may use your textbook and notes. It is due at the **beginning** of class on Wednesday.

Problem 1
(a) **(2 points).** *Compute the third order Taylor polynomial generated by* $f(x) = \sin x$ *at* $x = \pi/6$.

Solution: We compute the necessary derivatives.

$$\begin{aligned} f(\pi/6) &= \sin(\pi/6) = \frac{1}{2} \\ f'(\pi/6) &= \cos(\pi/6) = \frac{\sqrt{3}}{2} \\ f(\pi/6) &= -\sin(\pi/6) = -\frac{1}{2} \\ f(\pi/6) &= -\cos(\pi/6) = -\frac{\sqrt{3}}{2} \end{aligned}$$

Thus, the third order Taylor polynomial is given by

$$\begin{aligned} P_3(x) &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2}\frac{(x-\frac{\pi}{6})^2}{2!} - \frac{\sqrt{3}}{2}\frac{(x-\frac{\pi}{6})^3}{3!} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3 \end{aligned}$$

(b) **(1 point).** *Use your answer from part (a) to estimate* $\sin(31°)$

Hint: *You need to express* $31°$ *in radians in order to solve the problem.*

Solution:
We note that $31° = 30° + 1°$. In radians this is equal to $\pi/6 + \pi/180$.
Now we simply substitute $\pi/6 + \pi/180$ for x in $P_3(x)$.

$$\sin(31°) \approx P_3(\pi/6 + \pi/180) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right) - \frac{1}{4}\left(\frac{\pi}{180}\right)^2 - \frac{\sqrt{3}}{12}\left(\frac{\pi}{180}\right)^3$$

(c) **(2 points).** *Use the Remainder Estimation Theorem to esimate the magnitude of the error in the approximation from part (b).*

Solution:
We note that $f^{(4)}(x) = \sin x$ and that $|\sin x| \leq 1$ for all x.
Then, by the Remainder Estimation Theorem

$$|R_3(x)| \leq \frac{1}{4!}\left(\frac{\pi}{180}\right)^4 = \frac{\pi^4}{24(180)^4}$$

Problem 2 (1 point). *Use the fact that the following series is the value of the Maclaurin series of a function $f(x)$ at some point to find the sum of the series.*

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$$

Solution:
Recall that the Maclaurin series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x for all real numbers x. Therefore

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} = e^{-2}$$

Problem 3 (2 points). *Find the Maclaurin series for $f(x) = x^2 \cos(x^2)$*

Solution:
The Maclaurin series for $\cos x$ is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Therefore, the Maclaurin series for $\cos(x^2)$ is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{(2k)!}$$

Multiplying through by x^2 gives us that the Maclaurin series for $x^2 \cos(x^2)$ is

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x^2) x^{4k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k)!}$$

Problem 4 (2 points). *Let n be a fixed positive integer. Verify that the Maclaurin series generated by $f(x) = x^n$ is just x^n itself by direct calculation.*

Hint: *When calculating the derivatives $f^{(k)}(0)$, consider the cases $k < n$, $k > n$ and $k = n$ separately.*

Solution:
Calculating the k-th derivative of $f(x)$ yields

$$f^{(k)}(x) = \begin{cases} x^n & k = 0 \\ n(n-1)\cdots(n-k+1)x^{n-k} & 0 < k < n \\ n! & k = n \\ 0 & k > n \end{cases}$$

Evaluating at $x = 0$ yields

$$f^{(k)}(0) = \begin{cases} n! & k = n \\ 0 & k \neq n \end{cases}$$

Therefore the Maclaurin series generated by x^n is simply

$$\frac{n!}{n!}x^n = x^n$$