Complete each of the following problems. Show all work. You may use your textbook and notes. It is due at the beginning of class on Wednesday.

Problem 1

(a) (2 points). Compute the third order Taylor polynomial generated by \( f(x) = \sin x \) at \( x = \pi/6 \).

Solution: We compute the necessary derivatives.

\[
\begin{align*}
f(\pi/6) &= \sin(\pi/6) = \frac{1}{2} \\
f'(\pi/6) &= \cos(\pi/6) = \frac{\sqrt{3}}{2} \\
f(\pi/6) &= -\sin(\pi/6) = -\frac{1}{2} \\
f(\pi/6) &= -\cos(\pi/6) = -\frac{\sqrt{3}}{2}
\end{align*}
\]

Thus, the third order Taylor polynomial is given by

\[
P_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{6}\right) - \frac{1}{2} \left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12} \left(x - \frac{\pi}{6}\right)^3
\]

(b) (1 point). Use your answer from part (a) to estimate \( \sin(31^\circ) \)

Hint: You need to express 31° in radians in order to solve the problem.

Solution: We note that 31° = 30° + 1°. In radians this is equal to \( \pi/6 + \pi/180 \).
Now we simply substitute \( \pi/6 + \pi/180 \) for \( x \) in \( P_3(x) \).

\[
\sin(31^\circ) \approx P_3(\pi/6 + \pi/180) = \frac{1}{2} + \frac{\sqrt{3}}{2} \left(\frac{\pi}{180}\right) - \frac{1}{4} \left(\frac{\pi}{180}\right)^2 - \frac{\sqrt{3}}{12} \left(\frac{\pi}{180}\right)^3
\]
(c) (2 points). Use the Remainder Estimation Theorem to estimate the magnitude of the error in the approximation from part (b).

Solution:
We note that \( f^{(4)}(x) = \sin x \) and that \( |\sin x| \leq 1 \) for all \( x \).
Then, by the Remainder Estimation Theorem

\[
|R_3(x)| \leq \frac{1}{4!} \left( \frac{\pi}{180} \right)^4 = \frac{\pi^4}{24(180)^4}
\]

Problem 2 (1 point). Use the fact that the following series is the value of the Maclaurin series of a function \( f(x) \) at some point to find the sum of the series.

\[
\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}
\]

Solution:
Recall that the Maclaurin series \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) converges to \( e^x \) for all real numbers \( x \). Therefore

\[
\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} = e^{-2}
\]

Problem 3 (2 points). Find the Maclaurin series for \( f(x) = x^2 \cos(x^2) \)

Solution:
The Maclaurin series for \( \cos x \) is given by

\[
\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}
\]

Therefore, the Maclaurin series for \( \cos(x^2) \) is given by

\[
\sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{(2k)!}
\]

Multiplying through by \( x^2 \) gives us that the Maclaurin series for \( x^2 \cos(x^2) \) is

\[
\sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k)!}
\]
Problem 4 (2 points). Let \( n \) be a fixed positive integer. Verify that the Maclaurin series generated by \( f(x) = x^n \) is just \( x^n \) itself by direct calculation.

**Hint:** When calculating the derivatives \( f^{(k)}(0) \), consider the cases \( k < n \), \( k > n \) and \( k = n \) separately.

**Solution:**
Calculating the \( k \)-th derivative of \( f(x) \) yields

\[
f^{(k)}(x) = \begin{cases} 
  x^n & k = 0 \\
  n(n - 1) \cdots (n - k + 1)x^{n-k} & 0 < k < n \\
  n! & k = n \\
  0 & k > n 
\end{cases}
\]

Evaluating at \( x = 0 \) yields

\[
f^{(k)}(0) = \begin{cases} 
  n! & k = n \\
  0 & k \neq n 
\end{cases}
\]

Therefore the Maclaurin series generated by \( x^n \) is simply

\[
\frac{n!}{n!} x^n = x^n
\]