**Problem 1 (2 points).** *Solve the following differential equation.*

$$e^{y-x}\frac{dy}{dx} = x$$

**Solution:** We notice that the given equation is separable, and that multiplying both sides by $e^x dx$ yields

$$e^y\,dy = xe^x\,dx$$

The integral of the left hand side is just $e^y$. To integrate the right hand side, we use integration by parts with $u = x$ and $dv = e^x\,dx$.

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

Therefore the solution to the differential equation is $e^y = xe^x - e^x + C$.

**Problem 2.** *A tank initially contains 80 gal of brine in which 20 lb of salt are dissolved. A brine containing 2 lb/gal of salt runs into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring and flows out of the tank at a rate of 3 gal/min. Let $t$ denote time (in min) and $y$ denote the amount of salt in the tank (in lb).*

*(a)* **(1 point).** *What is the volume of brine in the tank at time $t$?*

**Solution:** There are initially 80 gal of brine in the tank. Brine flows into the tank at a constant rate of 5 gal/min and flows out at a constant rate of 3 gal/min. So, at time $t$, the volume of brine in the tank is $80 + 5t - 3t = 80 + 2t$ gal.

*(b)* **(3 points).** *Write down an initial value problem (a differential equation and an initial condition) describing the mixing process.*

**Note:** *You are only being asked to set up the problem, not to solve it.*

**Solution:**
Our differential equation should express the rate of change $y'$ of the amount of salt in the tank in terms of the time $t$ and the amount of salt $y$ in the tank at time $t$. The rate of change of the amount of salt in the tank is the rate at which salt is added to the tank minus the rate at which salt leaves the tank.

The rate at which salt enters the tank is the product of the rate at which the brine enters the tank and the concentration of salt in that brine. Therefore the rate at which salt enters the tank is 5 gal/min $\cdot 2$ lb/gal $= 10$ lb/min.

To find the rate at which salt leaves the tank, we'll need the concentration of salt in the brine. The concentration is the ratio of the amount of salt in the tank to the amount of brine in the tank. Therefore the concentration of salt in the brine is $\frac{y}{80+2t}$ lb/gal.

Now the rate at which salt leaves the tank is the product of the rate at which the brine leaves the tank and the concentration of salt in the brine. This gives us that the rate at which salt leaves the tank is 3 gal/min $\cdot \frac{y}{80+2t}$ lb/gal $= \frac{3y}{80+2t}$ lb/min.

Therefore, the differential equation describing the mixing process is $y' = 10 - \frac{3y}{80+2t}$.

Since there are initially 20 lb of salt in the tank, our initial condition is $y(0) = 20$.

**Problem 3 (4 points).** *Solve the following initial value problem for $y$ as a function of $x$.*

$$(x+1)\frac{dy}{dx} = 2y + x + 1,\ x > -1,\ y(0) = 0$$

**Solution:** This is a linear first order differential equation. Written in standard form, the equation becomes

$$\frac{dy}{dx} - \frac{2}{x+1}y = 1$$

So $P(x) = -\frac{2}{x+1}$ and therefore $\int P(x)\,dx = -\int \frac{2}{x+1}\,dx = -2\ln|x+1| = \ln(x+1)^{-2}$. Therefore, our integrating factor is

$$e^{\int P(x)\,dx} = e^{\ln(x+1)^{-2}} = (x+1)^{-2}$$

We now multiply both sides of the equation by the integrating factor and then integrate to find the general solution.

$$\begin{aligned}
(x+1)^{-2}\left(\frac{dy}{dx} - \frac{2}{x+1}y\right) &= (x+1)^{-2} \\
\int d\left[(x+1)^{-2}y\right] &= \int (x+1)^{-2}dx \\
(x+1)^{-2}y &= -(x+1)^{-1} + C \\
y &= -(x+1) + C(x+1)^2
\end{aligned}$$

Using our intial condition $y(0) = 0$ gives us $0 = -1 + C$ and so $C = 1$. Therefore, the particular solution is $y = (x+1)^2 - (x+1)$.