Problem 1 (3 points). *Solve the following initial value problem.*

$$y'' + 2y' - 3y = 0 \ , \ y(0) = 8 \ , \ y'(0) = 0$$

Solution: We first form the auxillary equation.

$$r^2 + 2r - 3 = 0 \Longrightarrow (r+3)(r-1) = 0$$

This gives us that $y(x) = c_1 e^x + c_2 e^{-3x}$. Taking the derivative gives us that $y'(x) = c_1 e^x - 3c_2 e^{-3x}$. Evaluating at $x = 0$ yields $y(0) = c_1 + c_2$ and $y'(0) = c_1 - 3c_2$. Applying our initial conditions ($y(0) = 8$ and $y'(0) = 0$) yields the two equations

$$\begin{aligned} c_1 + c_2 &= 8 \\ c_1 - 3c_2 &= 0 \end{aligned}$$

We now must solve the two equations simultaneously. For instance, the second equation gives us that $c_1 = 3c_2$ and plugging this into the first equation yields $4c_2 = 8$. Hence $c_2 = 2$. Plugging this into either equation yields $c_1 = 6$. So the solution is

$$y = 6e^x + 2e^{-3x}$$

Problem 2 (3 points). *Solve the following differential equation.*

$$2y'' - 2y' + y = x^2$$

Solution: To find the solution to the homogeneous equation, we form the auxillary equation $2r^2 - 2r + 1 = 0$ and then use the quadratic equation to solve for r.

$$r = \frac{2 \pm \sqrt{4 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{2 \pm \sqrt{-4}}{4} = \frac{1}{2} \pm \frac{1}{2} i$$

So, writing $r = \alpha \pm \beta i$, we get $\alpha = 1/2$ and $\beta = 1/2$. Hence the solution to the homogeneous equation is $y_h = e^{x/2}\left[c_1 \sin(x/2) + c_2 \cos(x/2)\right]$.

To find y_p we use the method of undetermined coefficients. Since the right hand side is x^2 (and 0 is not a root of the auxillary equation), our trial function will be the quadratic $y_p = Ax^2 + Bx + C$. Taking deriviatives yields $y_p' = 2Ax + B$ and $y_p'' = 2A$.

Now $2y_p'' - 2y_p' + y_p = 2(2A) - 2(2Ax + B) + Ax^2 + Bx + C = Ax^2 + (B - 4A)x + (4A - 2B + C)$. We want $2y_p'' - 2y_p' + y_p = x^2$, so equating coefficients yields $A = 1$, $B - 4A = 0$ and $4A - 2B + C = 0$. Since $A = 1$, we get that $B = 4$, which gives us that $C = 4$. So our solution is

$$y = e^{x/2}\left[c_1 \sin(x/2) + c_2 \cos(x/2)\right] + x^2 + 4x + 4$$

Problem 3 (4 points). *Solve the following differential equation.*

$$y'' - 4y' + 4y = e^{2x}$$

Solution: To find the solution to the homogeneous equation, we form the auxillary equation.

$$r^2 - 4r + 4 = 0 \Longrightarrow (r-2)^2 = 0$$

This gives us that $y_h = c_1 e^{2x} + c_2 x e^{2x}$. To find y_p we can use either the method of undetermined coefficents or variation of parameters.

Undetermined Coefficents. Our trial function for y_p must be linearly independent from e^{2x} and xe^{2x}, so we will use $y_p = Ax^2e^{2x}$. Then $y_p' = 2Ax^2e^{2x} + 2Axe^{2x}$ and $y_p'' = 4Ax^2e^{2x} + 8Axe^{2x} + 2Ae^{2x}$.

Now $y_p'' - 4y_p' + 4y_p = 4Ax^2e^{2x} + 8Axe^{2x} + 2Ae^{2x} - 4\left[2Ax^2e^{2x} + 2Axe^{2x}\right] + 4Ax^2e^{2x} = 2Ae^{2x}$. We want $y_p'' - 4y_p' + 4y_p = e^{2x}$, so equating coefficients yields $2A = 1$ and hence $A = 2$. Therefore $y_p = \frac{1}{2}x^2e^{2x}$.

Variation of Parameters. Let $y_1 = e^{2x}$ and $y_2 = xe^{2x}$. Our solution will be $y_p = v_1y_1 + v_2y_2$ where v_1 and v_2 satisfy $v_1'y_1 + v_2'y_2 = 0$ and $v_1'y_1' + v_2'y_2' = e^{2x}$. This gives us the two equations

$$\begin{aligned} v_1'e^{2x} + v_2'xe^{2x} &= 0 \\ v_1'2e^{2x} + v_2'(1+2x)e^{2x} &= e^{2x} \end{aligned}$$

The first equation gives us that $v_2'xe^{2x} = -v_1'e^{2x}$ and therefore $v_1' = -v_2'x$. Plugging this into the second equation yields $(-v_2x)2e^{2x} + v_2'(1+2x)e^{2x} = e^{2x}$ which, after simplifying, gives us $v_2'e^{2x} = e^{2x}$ and hence $v_2' = 1$. Therefore $v_1' = -v_2'x = -x$. Then $v_1 = \int v_1'\,dx = -\int x\,dx = -\frac{1}{2}x^2$ and $v_2 = \int v_2'\,dx = \int dx = x$. Therefore $y_p = -\frac{1}{2}x^2e^{2x} + x^2e^{2x} = \frac{1}{2}x^2e^{2x}$.

Therefore the solution to the differential equation is

$$y = c_1e^{2x} + c_2xe^{2x} + \frac{1}{2}x^2e^{2x}$$