Problem 1 (3 points). Solve the following initial value problem.

\[ y'' + 2y' - 3y = 0 , \ y(0) = 8 , \ y'(0) = 0 \]

Solution: We first form the auxiliary equation.

\[ r^2 + 2r - 3 = 0 \implies (r + 3)(r - 1) = 0 \]

This gives us that \( y(x) = c_1 e^x + c_2 e^{-3x} \). Taking the derivative gives us that \( y'(x) = c_1 e^x - 3c_2 e^{-3x} \). Evaluating at \( x = 0 \) yields \( y(0) = c_1 + c_2 \) and \( y'(0) = c_1 - 3c_2 \). Applying our initial conditions (\( y(0) = 8 \) and \( y'(0) = 0 \)) yields the two equations

\[
\begin{align*}
  c_1 + c_2 &= 8 \\
  c_1 - 3c_2 &= 0
\end{align*}
\]

We now must solve the two equations simultaneously. For instance, the second equation gives us that \( c_1 = 3c_2 \) and plugging this into the first equation yields \( 4c_2 = 8 \). Hence \( c_2 = 2 \). Plugging this into either equation yields \( c_1 = 6 \). So the solution is

\[ y = 6e^x + 2e^{-3x} \]

Problem 2 (3 points). Solve the following differential equation.

\[ 2y'' - 2y' + y = x^2 \]

Solution: To find the solution to the homogeneous equation, we form the auxiliary equation \( 2r^2 - 2r + 1 = 0 \) and then use the quadratic equation to solve for \( r \).

\[ r = \frac{2 \pm \sqrt{4 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{2 \pm \sqrt{-4}}{4} = \frac{1}{2} \pm \frac{1}{2}i \]

So, writing \( r = \alpha \pm \beta i \), we get \( \alpha = 1/2 \) and \( \beta = 1/2 \). Hence the solution to the homogeneous equation is \( y_h = e^{x/2} [c_1 \sin(x/2) + c_2 \cos(x/2)] \).

To find \( y_p \) we use the method of undetermined coefficients. Since the right hand side is \( x^2 \) (and 0 is not a root of the auxiliary equation), our trial function will be the quadratic \( y_p = Ax^2 + Bx + C \). Taking derivatives yields \( y'_p = 2Ax + B \) and \( y''_p = 2A \).

Now \( 2y''_p - 2y'_p + y_p = 2(2A) - 2(2Ax + B) + Ax^2 + Bx + C = Ax^2 + (B - 4A)x + (4A - 2B + C) \). We want \( 2y''_p - 2y'_p + y_p = x^2 \), so equating coefficients yields \( A = 1 \), \( B - 4A = 0 \) and \( 4A - 2B + C = 0 \). Since \( A = 1 \), we get that \( B = 4 \), which gives us that \( C = 4 \). So our solution is

\[ y = e^{x/2} [c_1 \sin(x/2) + c_2 \cos(x/2)] + x^2 + 4x + 4 \]
Problem 3 (4 points). Solve the following differential equation.

\[ y'' - 4y' + 4y = e^{2x} \]

Solution: To find the solution to the homogeneous equation, we form the auxiliary equation.

\[ r^2 - 4r + 4 = 0 \implies (r - 2)^2 = 0 \]

This gives us that \( y_h = c_1 e^{2x} + c_2 xe^{2x} \). To find \( y_p \) we can use either the method of undetermined coefficients or variation of parameters.

**Undetermined Coefficients.** Our trial function for \( y_p \) must be linearly independent from \( e^{2x} \) and \( xe^{2x} \), so we will use \( y_p = Ax^2 e^{2x} \). Then \( y_p' = 2Ax^2 e^{2x} + 2Axe^{2x} \) and \( y_p'' = 4Ax^2 e^{2x} + 8Axe^{2x} + 2Ae^{2x} \).

Now \( y_p'' - 4y_p' + 4y_p = 4Ax^2 e^{2x} + 8Axe^{2x} + 2Ae^{2x} - 4(2Ax^2 e^{2x} + 2Axe^{2x}) + 4Ax^2 e^{2x} = 2Ae^{2x} \).

We want \( y_p'' - 4y_p' + 4y_p = e^{2x} \), so equating coefficients yields \( 2A = 1 \) and hence \( A = 2 \). Therefore \( y_p = \frac{1}{2} x^2 e^{2x} \).

**Variation of Parameters.** Let \( y_1 = e^{2x} \) and \( y_2 = xe^{2x} \). Our solution will be \( y_p = v_1 y_1 + v_2 y_2 \), where \( v_1 \) and \( v_2 \) satisfy \( v_1' y_1 + v_2' y_2 = 0 \) and \( v_1' y_1' + v_2' y_2' = e^{2x} \). This gives us the two equations

\[
\begin{align*}
    v_1' e^{2x} + v_2' xe^{2x} &= 0 \\
    v_1' 2e^{2x} + v_2' (1 + 2x)e^{2x} &= e^{2x}
\end{align*}
\]

The first equation gives us that \( v_2' xe^{2x} = -v_1' e^{2x} \) and therefore \( v_1' = -v_2' x \). Plugging this into the second equation yields \( (-v_2 x)2e^{2x} + v_2' (1 + 2x)e^{2x} = e^{2x} \) which, after simplifying, gives us \( v_2' e^{2x} = e^{2x} \) and hence \( v_2' = 1 \). Therefore \( v_1' = -v_2' x = -x \).

Then \( v_1 = \int v_1' dx = -\int x dx = -\frac{1}{2} x^2 \) and \( v_2 = \int v_2' dx = \int dx = x \). Therefore \( y_p = -\frac{1}{2} x^2 e^{2x} + x^2 e^{2x} = \frac{1}{2} x^2 e^{2x} \).

Therefore the solution to the differential equation is

\[ y = c_1 e^{2x} + c_2 xe^{2x} + \frac{1}{2} x^2 e^{2x} \]