Problem 1  Consider the points $P_1(0, 2, 1)$ and $P_2(1, 4, -1)$

(a) (1 point). Find the direction of the vector $\overrightarrow{P_1P_2}$.

Solution: $\overrightarrow{P_1P_2}$ will be given by $\langle 1 - 0, 4 - 2, -1 - 1 \rangle = \langle 1, 2, -2 \rangle$.

The length of $\overrightarrow{P_1P_2}$ will then be $\sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$.

Therefore, the direction of $\overrightarrow{P_1P_2}$ is $\frac{1}{3} \langle 1, 2, -2 \rangle$.

(b) (1 point). Find the midpoint of the line segment $P_1P_2$.

Solution: The midpoint of the line segment is given by

$\left( \frac{0 + 1}{2}, \frac{2 + 4}{2}, \frac{1 - 1}{2} \right) = \left( \frac{1}{2}, 3, 0 \right)$

Problem 2 (3 points). Let $u = 2i + 3j - k$ and let $v = i - j + k$. Find the vector projection $\text{proj}_v u$.

Solution: We have that $\text{proj}_v u = \frac{u \cdot v}{|v|^2} v$.

Now $u \cdot v = 2 \cdot 1 + 3 \cdot (-1) + (-1) \cdot (1) = -2$.

Also $|v|^2 = 1^2 + (-1)^2 + 1^2 = 3$.

So, we have that $\text{proj}_v u$ is equal to

$$\frac{u \cdot v}{|v|^2} v = \frac{-2}{3} [i - j + k]$$

$$= \frac{-2}{3}i + \frac{2}{3}j - \frac{2}{3}k$$

Problem 3 (3 points). Find the center and radius of the sphere given by the equation $x^2 + y^2 + z^2 - 2x + 4y = -1$.

Solution: To find the center and radius, we must first put the equation in standard form. To do so we complete the square.

$$x^2 - 2x + y^2 + 4y + z^2 = -1$$

$$(x^2 - 2x + 1) - 1 + (y^2 + 4y + 4) - 4 + z^2 = -1$$

$$(x - 1)^2 + (y + 2)^2 + z^2 = 4$$

So the sphere has center $(1,-2,0)$ and radius 2.
Problem 4 (2 points). Let \( \mathbf{u} \) and \( \mathbf{v} \) be vectors such that \( |\mathbf{u}| = |\mathbf{v}| \). Use vector methods to show that the indicated diagonal of the parallelogram determined by \( \mathbf{u} \) and \( \mathbf{v} \) bisects the angle between \( \mathbf{u} \) and \( \mathbf{v} \).

Solution: We note that the diagonal is given by the vector \( \mathbf{u} + \mathbf{v} \). Then the cosine of the angle \( \alpha \) between \( \mathbf{v} \) and the diagonal \( \mathbf{u} + \mathbf{v} \) is given by

\[
\cos(\alpha) = \frac{(\mathbf{u} + \mathbf{v}) \cdot \mathbf{v}}{|\mathbf{u} + \mathbf{v}| |\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2}{|\mathbf{u} + \mathbf{v}| |\mathbf{v}|}.
\]

The cosine of the angle \( \beta \) between \( \mathbf{u} \) and the diagonal \( \mathbf{u} + \mathbf{v} \) is given by

\[
\cos(\beta) = \frac{(\mathbf{u} + \mathbf{v}) \cdot \mathbf{u}}{|\mathbf{u} + \mathbf{v}| |\mathbf{u}|} = \frac{\mathbf{u} \cdot \mathbf{v} + |\mathbf{u}|^2}{|\mathbf{u} + \mathbf{v}| |\mathbf{u}|}.
\]

But since \( |\mathbf{u}| = |\mathbf{v}| \), these two expressions are equal, and therefore \( \cos(\alpha) = \cos(\beta) \). Since \( \alpha \) and \( \beta \) are both between 0 and \( \pi/2 \), this implies that \( \alpha = \beta \).