Problem 1 *Consider the points $P_1(0,2,1)$ and $P_2(1,4,-1)$*

(a) **(1 point).** *Find the direction of the vector $\overrightarrow{P_1P_2}$.*

Solution: $\overrightarrow{P_1P_2}$ will be given by $\langle 1-0, 4-2, -1-1\rangle = \langle 1,2,-2\rangle$.
The length of $P_1\vec{P}_2$ will then be $\sqrt{1^2+2^2+(-2)^2} = \sqrt{9} = 3$.
Therefore, the direction of $\overrightarrow{P_1P_2}$ is $\frac{1}{3}\overrightarrow{P_1P_2} = \langle 1/3, 2/3, -2/3\rangle$.

(b) **(1 point).** *Find the midpoint of the line segment P_1P_2.*

Solution: The midpoint of the line segment is given by

$$\left(\frac{0+1}{2}, \frac{2+4}{2}, \frac{1-1}{2}\right) = \left(\frac{1}{2}, 3, 0\right)$$

Problem 2 (3 points). *Let* $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ *and let* $\mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k}$. *Find the vector projection* $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Solution: We have that $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$.
Now $\mathbf{u}\cdot\mathbf{v} = 2\cdot 1 + 3\cdot(-1) + (-1)\cdot(1) = -2$.
Also $|\mathbf{v}|^2 = 1^2 + (-1)^2 + 1^2 = 3$.
So, we have that $\text{proj}_{\mathbf{v}}\mathbf{u}$ is equal to

$$\begin{aligned}\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} &= \frac{-2}{3}[\mathbf{i} - \mathbf{j} + \mathbf{k}] \\ &= -\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\end{aligned}$$

Problem 3 (3 points). *Find the center and radius of the sphere given by the equation* $x^2 + y^2 + z^2 - 2x + 4y = -1$.

Solution: To find the center and radius, we must first put the equation in standard form. To do so we complete the square.

$$\begin{aligned}x^2 - 2x + y^2 + 4y + z^2 &= -1 \\ (x^2 - 2x + 1) - 1 + (y^2 + 4y + 4) - 4 + z^2 &= -1 \\ (x-1)^2 + (y+2)^2 + z^2 &= 4\end{aligned}$$

So the sphere has center (1,-2,0) and radius 2.

Problem 4 (2 points). *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors such that* $|\mathbf{u}| = |\mathbf{v}|$. *Use vector methods to show that the indicated diagonal of the parallelogram determined by* $\mathbf{u}$ *and* $\mathbf{v}$ *bisects the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.

Solution: We note that the diagonal is given by the vector $\mathbf{u}+\mathbf{v}$. Then the cosine of the angle α between $\mathbf{v}$ and the diagonal $\mathbf{u}+\mathbf{v}$ is given by

$$\cos(\alpha) = \frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{v}}{|\mathbf{u}+\mathbf{v}||\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}+|\mathbf{v}|^2}{|\mathbf{u}+\mathbf{v}||\mathbf{v}|}$$

The cosine of the angle β between $\mathbf{u}$ and the diagonal $\mathbf{u}+\mathbf{v}$ is given by

$$\cos(\beta) = \frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{u}}{|\mathbf{u}+\mathbf{v}||\mathbf{u}|} = \frac{\mathbf{u}\cdot\mathbf{v}+|\mathbf{u}|^2}{|\mathbf{u}+\mathbf{v}||\mathbf{u}|}$$

But since $|\mathbf{u}| = |\mathbf{v}|$, these two expressions are equal, and therefore $\cos(\alpha) = \cos(\beta)$. Since α and β are both between 0 and $\pi/2$, this implies that $\alpha = \beta$.