Problem 1 (3 points): Evaluate the following integral.

\[ \int (\tan x + \sin x)(\cot x + \csc x) \, dx \]

Solution: We multiply out, noting that \( \tan x \cot x = 1 \), \( \sin x \csc x = 1 \), \( \sin x \cot x = \cos x \) and \( \tan x \csc x = \sec x \):

\[ \int (\tan x + \sin x)(\cot x + \csc x) \, dx = \int (2 + \cos x + \sec x) \, dx = 2x + \sin x + \ln |\sec x + \tan x| + C \]

Problem 2 (3 points): Let \( k \) be a fixed positive integer. Show that

\[ \int_{0}^{\pi/2} \cos^{2k+1} \theta \, d\theta = \int_{0}^{1} (1 - u^2)^k \, du \]

Hint: Use a trigonometric identity and then a \( u \)-substitution.

Solution: We note that \( \cos^{2k+1} \theta = \cos \theta \cos^{2k} \theta \). Then, using the trigonometric identity \( \cos^2 \theta = 1 - \sin^2 \theta \) we obtain

\[ \int_{0}^{\pi/2} \cos^{2k+1} \theta \, d\theta = \int_{0}^{\pi/2} \cos \theta (1 - \sin^2 \theta)^k \, d\theta \]

Now take \( u = \sin \theta \). This gives us \( du = \cos \theta \, d\theta \). When \( \theta = 0, u = 0 \) and when \( \theta = \pi/2, u = 1 \), which gives us

\[ \int_{0}^{1} (1 - u^2)^k \, du \]

Problem 3 (4 points): Solve the following initial value problem for \( x \) as a function of \( t \).

\[ (t^2 + 1)(t + 1) \frac{dx}{dt} = 3t^2 + 3t + 2, \quad t > -1, \quad x(0) = 2 \]

Solution: Since \( t > -1 \), \( (t^2 + 1)(t + 1) \neq 0 \), so we can legally divide both sides by \( (t^2 + 1)(t + 1) \) to obtain

\[ \frac{dx}{dt} = \frac{3t^2 + 3t + 2}{(t^2 + 1)(t + 1)} \]

Then, multiplying both sides of the equation by \( dt \) and using relation \( dx = \frac{dx}{dt} \, dt \) we obtain

\[ dx = \frac{3t^2 + 3t + 2}{(t^2 + 1)(t + 1)} \, dt \]
Integrating the left hand side yields $x$. To integrate the right hand side, we perform a partial fraction decomposition.

$$\frac{3t^2 + 3t + 2}{(t^2 + 1)(t + 1)} = \frac{At + B}{(t^2 + 1)} + \frac{C}{(t + 1)}$$

Putting the right hand side over a common denominator and then equating the numerators on the both sides yields

$$3t^2 + 3t + 2 = (At + B)(t + 1) + C(t^2 + 1)$$

We now must solve for the undetermined coefficients.

**Method 1**

At $t = -1$, $t + 1 = 0$ and we’re left with $2 = 2C$, so $C = 1$. At $t = 0$, $At = 0$ and so we’re left with $2 = B + 1$, so $B = 1$. At $t = 1$, we obtain $8 = 2(A + 1) + 2$ so $6 = 2(A + 1)$, so $3 = A + 1$, and so $A = 2$.

**Method 2**

Multiplying out the right hand side yields $At^2 + At + Bt + B + Ct^2 + C$ and rearranging gives us $(A + C)t^2 + (A + B)t + (B + C)$. So we have

$$3t^2 + 3t + 2 = (A + C)t^2 + (A + B)t + (B + C)$$

This gives us three linear equations in three unknowns:

- $A + C = 3$
- $A + B = 3$
- $B + C = 2$

We then find $A$, $B$, and $C$ by solving the equations. For example, the second equation gives us $B = 3 - A$, so replacing $B$ with $3 - A$ in the third equation gives us $3 - A + C = 2$ or $C - A = -1$. Adding that to the first equation gives us $2C = 2$ and so $C = 1$. Plugging $C = 1$ into the first equation gives us $A = 2$ and plugging $C = 1$ into the third equation gives us $B = 1$.

Once we have the coefficients, we can integrate

$$\int \frac{3t^2 + 3t + 2}{(t^2 + 1)(t + 1)} \, dt = \int \left( \frac{2t + 1}{t^2 + 1} + \frac{1}{t + 1} \right) \, dt = \int \frac{2t}{t^2 + 1} \, dt + \int \frac{1}{t^2 + 1} \, dt + \int \frac{1}{t + 1} \, dt$$

For the first integral, we substitute $u = t^2 + 1$, and then $du = 2t \, dt$. The second integral simply equals $\tan^{-1}(t)$, and the third integral is equal to $\ln|t + 1|$, so we have

$$x = \ln|t^2 + 1| + \tan^{-1}(t) + \ln|t + 1| + C$$

Now, $t > -1$, so the absolute values are unnecessary. Furthermore, $x(0) = 2$, so we have $2 = 2\ln(1) + \tan^{-1}(0) + C$. Now $\ln(1) = \tan^{-1}(0) = 0$, so $C = 2$. This gives us a final answer of

$$x(t) = \ln(t^2 + 1) + \tan^{-1}(t) + \ln(t + 1) + 2$$