Problem 1 (3 points): *Evaluate the following integral.*

$$\int (\tan x + \sin x)(\cot x + \csc x)\, dx$$

Solution: We multiply out, noting that $\tan x \cot x = 1$, $\sin x \csc x = 1$, $\sin x \cot x = \cos x$ and $\tan x \csc x = \sec x$:

$$\int (\tan x + \sin x)(\cot x + \csc x)\, dx = \int (2 + \cos x + \sec x)\, dx =$$
$$2x + \sin x + \ln|\sec x + \tan x| + C$$

Problem 2 (3 points): *Let k be a fixed positive integer. Show that*

$$\int_0^{\pi/2} \cos^{2k+1}\theta\, d\theta = \int_0^1 (1-u^2)^k\, du$$

Hint: Use a trigonometric identity and then a u-substitution.

Solution: We note that $\cos^{2k+1}\theta = \cos\theta\cos^{2k}\theta$. Then, using the trigonometric identity $\cos^2\theta = 1 - \sin^2\theta$ we obtain

$$\int_0^{\pi/2} \cos^{2k+1}\theta\, d\theta = \int_0^{\pi/2} \cos\theta(1-\sin^2\theta)^k\, d\theta$$

Now take $u = \sin\theta$. This gives us $du = \cos\theta\, d\theta$. When $\theta = 0$, $u = 0$ and when $\theta = \frac{\pi}{2}$, $u = 1$, which gives us

$$\int_0^1 (1-u^2)^k\, du$$

Problem 3 (4 points): *Solve the following initial value problem for x as a function of t.*

$$(t^2+1)(t+1)\frac{dx}{dt} = 3t^2 + 3t + 2\ ,\ t > -1\ ,\ x(0) = 2$$

Solution: Since $t > -1$, $(t^2+1)(t+1) \neq 0$, so we can legally divide both sides by $(t^2+1)(t+1)$ to obtain

$$\frac{dx}{dt} = \frac{3t^2+3t+2}{(t^2+1)(t+1)}$$

Then, multiplying both sides of the equation by dt and using relation $dx = \frac{dx}{dt}dt$ we obtain

$$dx = \frac{3t^2+3t+2}{(t^2+1)(t+1)}\, dt$$

Integrating the left hand side yields x. To integrate the right hand side, we perform a partial fraction decomposition.

$$\frac{3t^2+3t+2}{(t^2+1)(t+1)} = \frac{At+B}{(t^2+1)} + \frac{C}{(t+1)}$$

Putting the right hand side over a common denominator and then equating the numerators on the both sides yields

$$3t^2+3t+2 = (At+B)(t+1) + C(t^2+1)$$

We now must solve for the undetermined coefficients.

Method 1

At $t=-1$, $t+1=0$ and we're left with $2=2C$, so $C=1$. At $t=0$, $At=0$ and so we're left with $2=B+1$, so $B=1$. At $t=1$, we obtain $8=2(A+1)+2$ so $6=2(A+1)$, so $3=A+1$, and so $A=2$.

Method 2

Multiplying out the right hand side yields $At^2+At+Bt+B+Ct^2+C$ and rearranging gives us $(A+C)t^2+(A+B)t+(B+C)$. So we have

$$3t^2+3t+2 = (A+C)t^2+(A+B)t+(B+C)$$

This gives us three linear equations in three unknowns:

$$A+C=3$$
$$A+B=3$$
$$B+C=2$$

We then find A, B, and C by solving the equations. For example, the second equation gives us $B=3-A$, so replacing B with $3-A$ in the third equation gives us $3-A+C=2$ or $C-A=-1$. Adding that to the first equation gives us $2C=2$ and so $C=1$. Plugging $C=1$ into the first equation gives us $A=2$ and plugging $C=1$ into the third equation gives us $B=1$.

Once we have the coefficients, we can integrate

$$\int \frac{3t^2+3t+2}{(t^2+1)(t+1)}\,dt = \int \left(\frac{2t+1}{t^2+1} + \frac{1}{t+1}\right) dt =$$
$$\int \frac{2t}{t^2+1}\,dt + \int \frac{1}{t^2+1}\,dt + \int \frac{1}{(t+1)}\,dt$$

For the first integral, we susbtitute $u=t^2+1$, and then $du=2t\,dt$. The second integral simply equals $\tan^{-1}(t)$, and the third integral is equal to $\ln|t+1|$, so we have

$$x = \ln|t^2+1| + \tan^{-1}(t) + \ln|t+1| + C$$

Now, $t>-1$, so the absolute values are unnecessary. Furthermore, $x(0)=2$, so we have $2 = 2\ln(1)+\tan^{-1}(0)+C$. Now $\ln(1)=\tan^{-1}(0)=0$, so $C=2$. This gives us a final answer of

$$x(t) = \ln(t^2+1) + \tan^{-1}(t) + \ln(t+1) + 2$$