MATH 222 QUIZ 2 SOLUTIONS FEBRUARY 7, 2007

Problem 1 (3 points): *Evaluate the following integral.*

$$\int \sin^6 x \cos^5 x \, dx$$

Solution: We have $\cos x$ to an odd power, so we can factor out one copy of $\cos x$ to obtain

$$\int \cos x \sin^6 x \cos^4 x \, dx = \int \cos x \sin^6 x (\cos^2 x)^2 \, dx =$$

$$\int \cos x \sin^6 x (1 - \sin^2 x)^2 \, dx$$

We now let $u = \sin x$. So $du = \cos x \, dx$ and we have

$$\int u^6 (1 - u^2)^2 \, du = \int u^6 (1 - 2u^2 + u^4) \, du =$$

$$\int u^6 - 2u^8 + u^{10} \, du = \frac{1}{7} u^7 - \frac{2}{9} u^9 + \frac{1}{11} u^{11} + C =$$

$$\frac{1}{7} \sin^7 x - \frac{2}{9} \sin^9 x + \frac{1}{11} \sin^{11} x + C$$

Problem 2 (3 points): *Evaluate the following integral:*

$$\int \frac{dx}{(1 - x^2)^{3/2}}$$

Solution: We make the substitution $x = \sin \theta$ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Therefore $dx = \cos \theta \, d\theta$ and our integral becomes

$$\int \frac{\cos \theta \, d\theta}{(1 - \sin^2 \theta)^{3/2}} = \int \frac{\cos \theta \, d\theta}{\cos^3 \theta} =$$

$$\int \sec^2 \theta \, d\theta = \tan \theta + C$$

If $\sin \theta = x$ then $\tan \theta = \frac{x}{\sqrt{1 - x^2}}$ (We can see this by drawing a right triangle with the length of the side opposite to θ equal to x and the length of the hypotenuse equal to 1. Then the length of the side adjacent to θ must be equal to $\sqrt{1 - x^2}$). Substituting this in to our result yields a final answer of

$$\frac{x}{\sqrt{1 - x^2}} + C$$

Problem 3 (3 points): *Evaluate the following integral.*

$$\int_0^1 \tan^{-1} x\, dx$$

Solution: Method 1: We integrate by parts. Let $u = \tan^{-1} x$ and let $dv = dx$, then $du = \frac{dx}{1+x^2}$ and $v = x$. Therefore

$$\int_0^1 \tan^{-1} x\, dx = \Big[x \tan^{-1} x\Big]_0^1 - \int_0^1 \frac{x\, dx}{1+x^2}$$

$$= \Big[x \tan^{-1} x - \frac{1}{2}\ln(x^2+1)\Big]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

Method 2: We let $y = \tan^{-1} x$ and use the formula $\int f^{-1}(x)\, dx = xf^{-1}(x) - \int f(y)\, dy$. When $x = 0, y = 0$ and when $x = 1, y = \frac{\pi}{4}$ Therefore

$$\int_0^1 \tan^{-1} x\, dx = \Big[x \tan^{-1} x\Big]_0^1 - \int_0^{\pi/4} \tan y\, dy =$$

$$\tan^{-1}(1) + \ln\cos\left(\frac{\pi}{4}\right) = \frac{\pi}{4} + \ln\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

Problem 4 (1 point): *Suppose f is a twice differentiable function such that $f(0) = 0$, $f'(1) = 0$ and*

$$\int_0^1 f(x)f''(x)\, dx = 0$$

Show that $f(x)$ is constant on [0,1]. **Hint: It is enough to show that $f'(x) = 0$ on the interval. Use integration by parts to establish this.**

Solution: We integrate by parts using $u = f(x)$ and $dv = f''(x)\, dx$. Then $du = f'(x)\, dx$ and $v = f'(x)$. Therefore we have

$$\int_0^1 f(x)f''(x)\, dx = \Big[f(x)f'(x)\Big]_0^1 - \int_0^1 [f'(x)]^2\, dx$$

Since the expression on the left is equal to zero, the expression on the right must also be equal to zero. So

$$\Big[f(x)f'(x)\Big]_0^1 - \int_0^1 [f'(x)]^2\, dx = 0$$

But since $f(0) = f'(1) = 0$, we have

$$\Big[f(x)f'(x)\Big]_0^1 = f(1)f'(1) - f(0)f'(0) = (f(1)\cdot 0) - (0\cdot f'(0)) = 0$$

This means that $0 - \int_0^1 [f'(x)]^2\, dx = 0$ which gives us that $\int_0^1 [f'(x)]^2\, dx = 0$. Since $[f'(x)]^2 \geq 0$ the only way that this could happen is if $[f'(x)]^2 = 0$ on [0,1]. Therefore $f'(x) = 0$, which means that f is constant on [0,1] (in fact, since we know that $f(0) = 0$, we have that $f(x) = 0$ on [0,1]).