Problem 1. *Consider the conic section given by the following equation.*

$$2x^2 + y^2 = 4$$

(a) **(2 points).** *Sketch the conic section, labeling all foci and vertices*

Solution: Putting the equation in standard form yields

$$\frac{x^2}{2} + \frac{y^2}{4} = 1$$

So $a = 2$, $b = \sqrt{2}$ and $c = \sqrt{4-2} = \sqrt{2}$. The foci are located at $(0, \pm\sqrt{2})$ and the vertices are located at $(0, \pm 2)$. Your sketch should include all of this.
(b) **(1 point).** *Find an equation for the tangent to the conic at the point* $(1, \sqrt{2})$

Solution: Implicit differentiation yields

$$4x + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$ gives us

$$\frac{dy}{dx} = \frac{-2x}{y}$$

So, at $(1, \sqrt{2})$ the slope of the tangent line is $-\frac{2}{\sqrt{2}} = -\sqrt{2}$. Then, by the point slope equation

$$y - \sqrt{2} = -\sqrt{2}(x-1)$$

Problem 2 (3 points). *Find an equation for the hyperbola with foci at* $(-4, 0)$ *and* $(6, 0)$ *and with vertices at* $(-2, 0)$ *and* $(4, 0)$.

Solution: Since the foci are equidistant from the center, we have that the hyperbola is centered at $(\frac{-4+6}{2}, 0) = (1, 0)$. The center to focus distance is then $c = 6 - 1 = 5$. We also have $a = 4 - 1 = 3$. Then since $c^2 = a^2 + b^2$, we have $b^2 = c^2 - a^2 = 25 - 9 = 16$, so $b = 4$. Then the equation of the hyperbola is

$$\frac{(x-1)^2}{9} - \frac{y^2}{16} = 1$$

Problem 3 (3 points). *Evaluate the following improper integral.*

$$\int_0^\infty xe^{-x}\,dx$$

Solution: We integrate by parts, letting $u = x$ and $dv = e^{-x}\,dx$. Then $du = dx$ and $v = -e^{-x}$. And so

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx =$$

$$\lim_{b\to\infty}\left(\left[-xe^{-x}\right]_0^b + \int_0^b e^{-x}\,dx\right) =$$

$$\lim_{b\to\infty}\left[-xe^{-x}-e^{-x}\right]_0^b = \lim_{b\to\infty}\left[\frac{-b-1}{e^b}\right]-(-1)$$

By L'Hôpital's Rule,

$$\lim_{b\to\infty}\left[\frac{-b-1}{e^b}\right] = \lim_{b\to\infty}\left[\frac{-1}{e^b}\right] = 0$$

So

$$\int_0^\infty xe^{-x}\,dx = 1$$

Problem 4 (1 point). *Let f be a function such that F is an antiderivative of f and*

$$\lim_{x\to\infty} F(x) = 0 \text{ and } \lim_{x\to-\infty} F(x) = 0$$

Is it necessarily true that $\int_{-\infty}^{\infty} f(x)\,dx$ converges? Justify your answer.

Solution: It is *not* necessarily true that $\int_{-\infty}^{\infty} f(x)\,dx$ converges, since we aren't told anything about whether f has any discontinuities within the interval. For example, consider the integral

$$\int_0^\infty \frac{dx}{x^2}$$

$\frac{-1}{x}$ is an antiderivative of f, and

$$\lim_{x\to\infty}\frac{-1}{x} = 0 \text{ and } \lim_{x\to-\infty}\frac{-1}{x} = 0$$

$$\text{But } \int_{-1}^{1}\frac{dx}{x^2} = \int_{-1}^{0}\frac{dx}{x^2} + \int_{0}^{1}\frac{dx}{x^2} =$$

$$\lim_{b\to 0}\int_{-1}^{b}\frac{dx}{x^2} + \lim_{c\to 0}\int_{c}^{1}\frac{dx}{x^2} =$$

$$\lim_{b\to 0}\left[\frac{-1}{x}\right]_{-1}^{b} + \lim_{c\to 0}\left[\frac{-1}{x}\right]_{c}^{1} =$$

$$\lim_{b\to 0}\frac{-1}{b} - 1 - 1 - \lim_{c\to 0}\frac{-1}{c}$$

$$\text{Then since } \lim_{b\to 0}\frac{-1}{b} = -\infty, \text{ we have that } \int_{-1}^{1}\frac{dx}{x^2} \text{ diverges}$$

$$\text{Therefore } \int_{-\infty}^{\infty}\frac{dx}{x^2} \text{ diverges even though } \lim_{x\to\infty}\frac{-1}{x} = 0 \text{ and } \lim_{x\to-\infty}\frac{-1}{x} = 0$$