

**Problem 1 (3 points).** Find  $e$ ,  $k$  and  $a$  for the conic section given by

$$r = \frac{8}{6 + 3 \cos \theta}$$

**Solution:** To find  $e$  and  $k$  we need to get the formula into the form

$$r = \frac{ke}{1 + e \cos \theta}$$

We find that

$$r = \frac{8}{6(1 + (1/2) \cos \theta)} = \frac{(8/3)(1/2)}{1 + (1/2) \cos \theta}$$

So  $k = \frac{8}{3}$  and  $e = \frac{1}{2}$ . Since  $e < 1$ , we have an ellipse, and we can therefore use the equation  $k = \frac{a}{e} - ae$  to find  $a$ . So  $ke = a(1 - e^2)$ , so

$$\left(\frac{1}{2}\right)\left(\frac{8}{3}\right) = a\left(1 - \frac{1}{4}\right)$$

$$\text{Therefore } \frac{4}{3} = \frac{3}{4}a, \text{ so } a = \frac{16}{9}$$

**Problem 2.** Consider the conic section given by the equation

$$xy + 2x - 2y + 7 = 0$$

(a) **(2 points).** Rotate the coordinate axes to transform the equation into one with no cross product ( $xy$ ) term.

**Solution:**

$$\cot(2\alpha) = \frac{A - C}{B} = \frac{0 - 0}{1} = 0$$

So  $2\alpha = \frac{\pi}{2}$  and therefore  $\alpha = \frac{\pi}{4}$ . Then since  $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$  we have

$$x = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y' \text{ and } y = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$$

Substituting into the given equation yields

$$\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) + 2\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right) - 2\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) + 7 = 0$$

$$\text{which reduces to } \frac{1}{2}(x')^2 - \frac{1}{2}(y')^2 - 2\sqrt{2}y' + 7 = 0$$

(b) **(1 point).** Use the discriminant test to decide whether the equation represents a hyperbola, a parabola, or an ellipse.

**Solution:** We have that  $A = 0$ ,  $B = 1$  and  $C = 0$ , so by the discriminant test

$$B^2 - 4AC = 1 - 4 \cdot 0 \cdot 0 = 1 > 0$$

Therefore the equation represents a hyperbola.

**Problem 3.** Consider the polar equation

$$r^2 \cos(2\theta) = 1$$

(a) **(2 points).** Replace the equation by an equivalent Cartesian equation.

**Solution:** We use the trigonometric identity  $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$ . So  $r^2 \cos(2\theta) = r^2 \cos^2 \theta - r^2 \sin^2 \theta = (r \cos \theta)^2 - (r \sin \theta)^2$ . Then since  $x = r \cos \theta$  and  $y = r \sin \theta$  the equation becomes

$$x^2 - y^2 = 1$$

(b) **(1 point).** What is the eccentricity of the conic section given by this equation?

**Hint:** Use your answer from part (a)

**Solution:** We have a hyperbola with  $a = 1$  and  $b = 1$ . Then  $c = \sqrt{a^2 + b^2} = \sqrt{2}$ . Therefore the eccentricity is given by

$$e = \frac{c}{a} = \sqrt{2}$$

**Problem 4 (1 points).** Suppose a point is given by the polar coordinates  $(r, \theta)$ , where  $r \neq 0$ . Find **all** the polar coordinates of the point.

**Solution:** Adding an even multiple of  $\pi$  to  $\theta$  will also give polar coordinates of the point. Likewise, replacing  $r$  by  $-r$  and adding an odd multiple of  $\pi$  to  $\theta$  will also give polar coordinates of the point. These are the only possibilities. Therefore, all polar coordinates of the point are given by

$(r, \theta + 2k\pi)$  where  $k$  is an integer, and  $(-r, \theta + (2k + 1)\pi)$  where  $k$  is an integer.