Problem 1 (3 points). *Find the points of intersection of the following pair of curves.*

$$r = \cos\theta \text{ and } r = 1 - \cos\theta$$

Solution: We solve the two equations simultaneously.

$$\begin{aligned} \cos\theta &= 1 - \cos\theta \\ 2\cos\theta &= 1 \\ \cos\theta &= \frac{1}{2} \end{aligned}$$

$$\theta = \frac{\pi}{3} \text{ or } \theta = \frac{5\pi}{3}$$

So we have that the curves intersect at $(1/2, \pi/3)$ and $(1/2, 5\pi/3)$.
If we graph the two equations together, we notice that they also intersect at the origin. Putting this all together, the points of intersection are

$$(0,0), (1/2, \pi/3) \text{ and } (1/2, 5\pi/3)$$

Problem 2 (3 points). *Find the length of the curve $r = \sec\theta$ between $\theta = \pi/6$ and $\theta = \pi/3$.*

Solution: If $r = \sec\theta$ then $r^2 = \sec^2\theta$. Also, $\frac{dr}{d\theta} = \sec\theta\tan\theta$. The formula for arc length gives us

$$L = \int_{\pi/6}^{\pi/3} \sqrt{\sec^2\theta + \sec^2\theta\tan^2\theta}\, d\theta = \int_{\pi/6}^{\pi/3} \sqrt{\sec^2\theta(1+\tan^2\theta)}\, d\theta =$$

Using the trigonometric identity $\sec^2\theta = 1 + \tan^2\theta$ gives us

$$L = \int_{\pi/6}^{\pi/3} \sqrt{\sec^4\theta}\, d\theta = \int_{\pi/6}^{\pi/3} \sec^2\theta\, d\theta = \Big[\tan\theta\Big]_{\pi/6}^{\pi/3} = \sqrt{3} - \frac{1}{\sqrt{3}}$$

Problem 3 (3 points). *Find the area outside the curve $r = 1$ and inside the curve $r = 2\sin\theta$.*

Solution: We first find the points of intersection by setting the two equations equal. Doing so yields

$$\begin{aligned} 1 &= 2\sin\theta \\ \frac{1}{2} &= \sin\theta \end{aligned}$$

$$\theta = \frac{\pi}{6} \text{ or } \theta = \frac{5\pi}{6}$$

Graphing the two curves together, we see that there are no other intersections, and that for $\pi/6 \leq \theta \leq 5\pi/6$, we have $2\sin\theta \geq 1$. Also, we note that the area to the right of the y-axis ($\theta = \pi/2$) is equal to the area to the left of the y-axis. Therefore, letting $r_1 = 1$ and $r_2 = 2\sin\theta$ we have that the desired area is given by

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} {r_2}^2 - {r_1}^2 \, d\theta = 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} ({r_2}^2 - {r_1}^2)\, d\theta\right] = \int_{\pi/6}^{\pi/2} (4\sin^2\theta - 1)\, d\theta =$$

$$\int_{\pi/6}^{\pi/2} \left[4\left(\frac{1-\cos(2\theta)}{2}\right) - 1\right] d\theta = \int_{\pi/6}^{\pi/2} 1 - 2\cos(2\theta)\, d\theta =$$

$$\left[\theta - \sin(2\theta)\right]_{\pi/6}^{\pi/2} =$$

$$\frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

Problem 4 (1 point). *Identify the symmetries of the curve $r = \theta^2$.*

Solution: Replacing r and θ by r and $-\theta$ yields $r = (-\theta)^2 = \theta^2$, which is our original equation. So the curve is symmetric about the x-axis.

Replacing r and θ by r $\pi - \theta$ yields $r = (\pi - \theta)^2 = \pi^2 - 2\theta + \theta^2$, which is not our original equation. Replacing r and θ by $-r$ and $-\theta$ yields $-r = (-\theta)^2 = \theta^2$, which is not our original equation. So the curve is not symmetric about the y-axis.

It is impossible for a curve to possess two of the symmetries but not the third, so the curve is not symmetric about the origin (we could also see this by making the appropriate substitutions for r and θ and seeing that we do not get our original equation back).