Problem 1
(a) **(1 point).** *Use the discriminant test to decide whether the equation* $-2x^2 - xy + y^2 + 7x - 8y + 15 = 0$ *represents an ellipse, a parabola, or a hyperbola.*

Solution: $A = -2$, $-B = 1$ and $C = 1$ so the discriminant is

$$B^2 - 4AC = 1 - (4)(-2)(1) = 9 > 0$$

This implies that the equation represents a hyperbola.
(b) **(2 points).** *Show that the graph of* $-2x^2 - xy + y^2 + 7x - 8y + 15 = 0$ *is actually the two lines* $y = 2x + 3$ *and* $y = -x + 5$.

Solution: $y = 2x + 3$ is equivalent to $y - 2x - 3 = 0$, and $y = -x + 5$ is equivalent to $y + x - 5 = 0$.

$$(y - 2x - 3)(y + x - 5) = y^2 + xy - 5y - 2xy - 2x^2 + 10x - 3y - 3x + 15 =$$

$$-2x^2 - xy + y^2 + 7x - 8y + 15$$

$$\text{So } -2x^2 - xy + y^2 + 7x - 8y + 15 = 0 \text{ factors into } (y - 2x - 3)(y + x - 5) = 0$$

Therefore the graph of $-2x^2 - xy + y^2 + 7x - 8y + 15 = 0$ is the two lines $y = 2x + 3$ and $y = -x + 5$.
Problem 2 (2 points). *Evaluate the following integral.*

$$\int_0^{\pi} \sqrt{1 - \cos x}\, dx$$

Solution: We use the identity $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$. This allows us to replace $1 - \cos x$ with $2\sin^2(x/2)$. So

$$\int_0^{\pi} \sqrt{1 - \cos x}\, dx = \int_0^{\pi} \sqrt{2\sin^2\left(\frac{x}{2}\right)}\, dx = \sqrt{2}\int_0^{\pi} |\sin\left(\frac{x}{2}\right)|\, dx$$

Since $\sin(\frac{x}{2}) \geq 0$ on $[0, \pi]$ we have

$$\int_0^{\pi} \sqrt{1 - \cos x}\, dx = \int_0^{\pi} \sin\left(\frac{x}{2}\right) dx = \left[-2\sqrt{2}\cos\left(\frac{x}{2}\right)\right]_0^{\pi} = 2\sqrt{2}$$

Problem 3 (2 points). *Determine whether the following improper integral converges or diverges.*

$$\int_1^{\infty} \frac{\cos^2 x\, dx}{x^2 + 1}$$

Hint: *You only need to show whether or not the integral converges. Do not attempt to actually compute it.*

Solution: Since $-1 \leq \cos x \geq 1$ for all real numbers, we have that $0 \leq \cos^2 x \leq 1$ for all real numbers, and therefore

$$0 \leq \frac{\cos^2 x}{x^2+1} \leq \frac{1}{x^2+1}$$

This gives us that

$$0 \leq \int_1^\infty \frac{\cos^2 x \, dx}{x^2+1} \leq \int_1^\infty \frac{dx}{x^2+1}$$

We can compute this new integral directly.

$$\int_1^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2+1} =$$

$$\lim_{b\to\infty} \Big[\arctan x\Big]_1^b = \lim_{b\to\infty} \arctan b - \arctan 1 =$$

$$\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty$$

$$\text{So } \int_1^\infty \frac{dx}{x^2+1} \text{ converges.}$$

Therefore, by the Direct Comparison Test

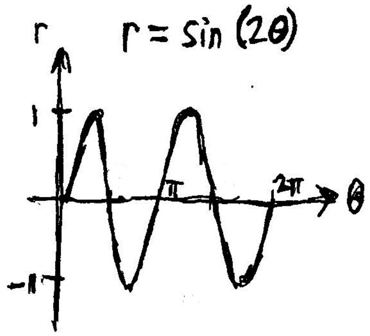
$$\int_1^\infty \frac{\cos^2 x \, dx}{x^2+1} \text{ converges.}$$

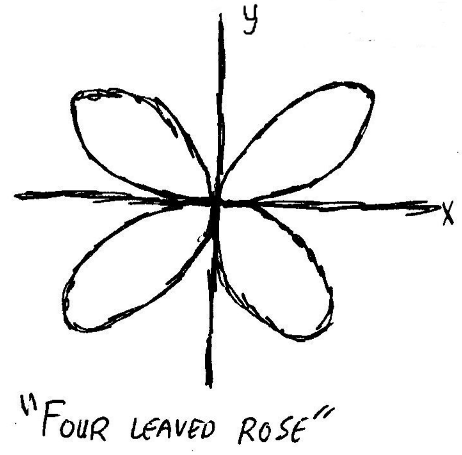
Problem 4

(a) **(1 point).** *Sketch a graph of*

$$r = \sin(2\theta)$$

Solution:
Your graph should resemble the following. The $r\theta$ graph is on the left and the xy graph (i.e. the final answer) is on the right.





(b) **(2 points).** *Find the equation of the tangent line to* $r = \sin(2\theta)$ *at* $\theta = \frac{\pi}{4}$

Solution:
$r = f(\theta) = \sin(2\theta)$ so $f'(\theta) = 2\cos(2\theta)$. We apply the formula

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$= \frac{2\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta}{2\cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta}$$

Evaluating at $\theta = \pi/4$ gives us that the slope of the tangent line is

$$m = \frac{2\cos(\pi/2)\sin(\pi/4) + \sin(\pi/2)\cos(\pi/4)}{2\cos(\pi/2)\cos(\pi/4) - \sin(\pi/2)\sin(\pi/4)}$$

$$= \frac{0 + 1\cdot\frac{\sqrt{2}}{2}}{0 - 1\cdot\frac{\sqrt{2}}{2}}$$

$$= -1$$

When $\theta = \pi/4$, $r = \sin(\pi/2) = 1$ so $x = \cos(\pi/4) = \sqrt{2}/2$ and $y = \sin(\pi/4) = \sqrt{2}/2$. Therefore, the equation of the tangent line is

$$y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)$$