Problem 1 (2 points each). *Determine whether each of the following series converge or diverge. Be sure to clearly indicate which convergence test you're using for each part.*

(a) $\sum_{n=1}^{\infty} \frac{n!}{n2^n}$

Solution: We apply the ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{(n+1)!}{(n+1)2^{n+1}} \cdot \frac{n2^n}{n!} \right] = \lim_{n\to\infty} \frac{n}{2} = \infty > 1$$

Therefore, the series diverges.

(b) $\sum_{n=1}^{\infty} \left(\frac{3n+1}{5n-2}\right)^n$

Solution:
We apply the root test.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{3n+1}{5n-2}\right)^n} = \lim_{n\to\infty} \left(\frac{3n+1}{5n-2}\right) = \frac{3}{5} < 1$$

Therefore, the series converges.

(c) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+5}$

We apply the limit comparison test with $\sum \frac{1}{n}$. Let $a_n = \frac{n^2}{n^3+5}$ and $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^3+5} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3}{n^3+5} = 1$$

Since $\sum \frac{1}{n}$ diverges, $\sum \frac{n^2}{n^3+5}$ diverges by the limit comparison test.

Problem 2 (4 points). *Determine whether the following series converges absolutely, converges conditionally, or diverges.*

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}-1}$$

Solution: Letting $u_n = \frac{1}{\sqrt{n}-1}$, we see that

$$u_n \geq 0 \text{ for all } n$$

$$u_n \geq u_{n+1} \text{ for all } n$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0$$

Therefore, by the alternating series test, the series converges.

Now we investigate the convergence of $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}-1}\right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$.

Note that $0 \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n}-1}$. But $\sum_{2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{2}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series ($p = \frac{1}{2} \leq 1$), so by the comparison test, $\sum_{2}^{\infty} \frac{1}{\sqrt{n}-1}$ also diverges.

Therefore, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}-1}$ converges conditionally.