Problem 1  Consider the following power series

$$
\sum_{n=1}^{\infty} \frac{(3x + 2)^n}{n2^n}
$$

(a) (2 points).  Find the radius of convergence of the series.

Solution:  We apply the ratio test.  Let $a_n = \frac{(3x + 2)^n}{n2^n}$.

$$
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{|3x + 2|^{n+1}}{(n + 1)2^{n+1}} \frac{n2^n}{n2^n} = \frac{|3x + 2|}{2} \lim_{n \to \infty} \frac{n}{n+1} = \frac{|3x + 2|}{2}
$$

So the series converges absolutely whenever $-1 < \frac{1}{2}(3x + 2) < 1$.  Multiplying by 2 throughout yields $-2 < 3x + 2 < 2$.  Subtracting 2 throughout yields $-4 < 3x < 0$.  Finally, dividing by 3 throughout yields $-4/3 < x < 0$.  So the interval of convergence has length $4/3$, and therefore the radius of convergence is $2/3$.

(b) (2 points).  Find the interval of convergence of the series.  Be sure to clearly indicate whether or not the series converges at each endpoint.

Solution:

We know the series converges absolutely for $-4/3 < x < 0$.  We also know the series diverges for $x < -4/3$ and for $x > 0$.  We only need to check the endpoints.  For $x = -4/3$, the series becomes

$$
\sum_{n=1}^{\infty} \frac{3(-\frac{4}{3}) + 2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}
$$

This series converges by the Alternating Series Test.  For $x = 0$, the series becomes

$$
\sum_{n=1}^{\infty} \frac{3(0) + 2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}
$$

This is the Harmonic Series, which diverges.

So the series converges on the interval $-4/3 \leq x < 0$.  

Problem 2

(a) (2 points). Consider the following series.

\[ \sum_{n=0}^{\infty} (-1)^n (2x)^n \]

Find the interval of convergence and, within this interval, the sum of the series as a function of \( x \).

Solution: This is a geometric series with \( r = -2x \). Therefore it converges if and only if \( | -2x | < 1 \), i.e., whenever \( |x| < 1/2 \). So the series converges for \(-1/2 < x < 1/2\). For those values of \( x \), we have that

\[ \sum_{n=0}^{\infty} (-1)^n (2x)^n = \frac{1}{1 - (-2x)} = \frac{1}{1 + 2x} \]

(b) (2 points). Consider the following series.

\[ \sum_{n=1}^{\infty} (-1)^n 2^n n x^{n-1} \]

Use your answer from part (a) to find the sum of the series as a function of \( x \).

Solution:
Notice that \( \frac{d}{dx}((-1)^n (2x)^n) = (-1)^n 2^n n x^{n-1} \). Therefore, we can apply the Term-by-Term Differentiation Theorem to conclude that the series converges for \(-1/2 < x < 1/2\), and that for those values of \( x \)

\[ \sum_{n=1}^{\infty} (-1)^n 2^n n x^{n-1} = \frac{d}{dx} \left( \frac{1}{1 + 2x} \right) = \frac{-2}{(1 + 2x)^2} \]

Problem 3 (2 points).

Find the first three terms of a power series that converges to \( e^x \) whenever \(-1 < x < 1\).

Solution:
The series \( \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots \) converges to \( e^x \) for all \( x \). We also have that \( \sum_{n=0}^{\infty} x^n \) converges to \( \frac{1}{1-x} \) whenever \(-1 < x < 1\).

Now let \( a_n = 1 \), \( b_n = \frac{1}{n!} \) and \( c_n = \sum_{k=0}^{n} a_k b_{n-k} \). By the Series Multiplication Theorem for Power Series, the series \( \sum_{n=0}^{\infty} c_n x^n \) converges absolutely to \( \frac{e^x}{1-x} \).

Now \( c_0 = a_0 b_0 = 1 \cdot 1 = 1 \), \( c_1 = a_0 b_1 + a_1 b_0 = 1 \cdot 1 + 1 \cdot 1 = 2 \) and \( c = 2 = a_0 b_2 + a_1 b_1 + a_2 b_0 = 1 \cdot \frac{1}{2!} + 1 \cdot 1 + 1 \cdot 1 = 2 + \frac{1}{2} = \frac{5}{2} \).

So, we have that \( \frac{e^x}{1-x} = 1 + 2x + \frac{5}{2} x^2 + \ldots \).