

NAME:

Answer each of the following problems. Show all work. Unsupported answers will receive little to no credit. Write legibly and clearly label your answer. Any work that is crossed out will not be assessed.

Problem 1 Consider the following power series

$$\sum_{n=1}^{\infty} \frac{(3x+2)^n}{n2^n}$$

(a) (2 points). Find the radius of convergence of the series.

Solution: We apply the ratio test. Let $a_n = \frac{(3x+2)^n}{n2^n}$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{|3x+2|^{n+1}}{(n+1)2^{n+1}} \frac{n2^n}{|3x+2|^n} \\ &= \frac{|3x+2|}{2} \lim_{n \rightarrow \infty} \frac{n}{n+1} \\ &= \frac{|3x+2|}{2} \end{aligned}$$

So the series converges absolutely whenever $-1 < (1/2)(3x+2) < 1$. Multiplying by 2 throughout yields $-2 < 3x+2 < 2$. Subtracting 2 throughout yields $-4 < 3x < 0$. Finally, dividing by 3 throughout yields $-4/3 < x < 0$. So the interval of convergence has length $4/3$, and therefore the radius of convergence is $2/3$.

(b) (2 points). Find the interval of convergence of the series. Be sure to clearly indicate whether or not the series converges at each endpoint.

Solution:

We know the series converges absolutely for $-4/3 < x < 0$. We also know the series diverges for $x < -4/3$ and for $x > 0$. We only need to check the endpoints. For $x = -4/3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{[3(-\frac{4}{3})+2]^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This series converges by the Alternating Series Test. For $x = 0$, the series becomes

$$\sum_{n=1}^{\infty} \frac{[3(0)+2]^n}{n2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

This is the Harmonic Series, which diverges.

So the series converges on the interval $-4/3 \leq x < 0$.

Problem 2

(a) (2 points). Consider the following series.

$$\sum_{n=0}^{\infty} (-1)^n (2x)^n$$

Find the interval of convergence and, within this interval, the sum of the series as a function of x .

Solution: This is a geometric series with $r = -2x$. Therefore it converges if and only if $|-2x| < 1$, i.e., whenever $|x| < 1/2$. So the series converges for $-1/2 < x < 1/2$. For those values of x , we have that

$$\sum_{n=0}^{\infty} (-1)^n (2x)^n = \frac{1}{1 - (-2x)} = \frac{1}{1 + 2x}$$

(b) (2 points). Consider the following series.

$$\sum_{n=1}^{\infty} (-1)^n 2^n n x^{n-1}$$

Use your answer from part (a) to find the sum of the series as a function of x .

Solution:

Notice that $\frac{d}{dx}((-1)^n (2x)^n) = (-1)^n 2^n n x^{n-1}$. Therefore, we can apply the Term-by-Term Differentiation Theorem to conclude that the series converges for $-1/2 < x < 1/2$, and that for those values of x

$$\sum_{n=1}^{\infty} (-1)^n 2^n n x^{n-1} = \frac{d}{dx} \left(\frac{1}{1 + 2x} \right) = \frac{-2}{(1 + 2x)^2}$$

Problem 3 (2 points).

Find the first three terms of a power series that converges to $\frac{e^x}{1-x}$ whenever $-1 < x < 1$.

Solution:

The series $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots$ converges to e^x for all x . We also have that $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ whenever $-1 < x < 1$.

Now let $a_n = 1$, $b_n = \frac{1}{n!}$ and $c_n = \sum_{k=0}^n a_n b_{n-k}$. By the Series Multiplication Theorem for Power Series, the series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $\frac{e^x}{1-x}$.

Now $c_0 = a_0 b_0 = 1 \cdot 1 = 1$, $c_1 = a_0 b_1 + a_1 b_0 = 1 \cdot 1 + 1 \cdot 1 = 2$ and $c_2 = 2 = a_0 b_2 + a_1 b_1 + a_2 b_0 = 1 \cdot \frac{1}{2!} + 1 \cdot 1 + 1 \cdot 1 = 2 + \frac{1}{2} = \frac{5}{2}$.

So, we have that $\frac{e^x}{1-x} = 1 + 2x + \frac{5}{2}x^2 + \dots$