

Math 319 Solutions to Review for Exam 1

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Problem 1.

(a) We note that $\frac{dy}{dx} = -\frac{1}{x}y + \cos x$ and so the ODE is linear. Converting to standard form gives $y' + \frac{1}{x}y = \cos x$. We can then compute that the integrating factor is $\mu(x) = x$. The problem then reduces down to $xy = \int x \cos x dx$. The integral can be computed using integration by parts with $u = x$ and $dv = \cos x$. We will end up with a solution of

$$y = \sin x + \frac{\cos x}{x} + \frac{C}{x}$$

(b) This equation is neither linear nor separable, so our only hope of solving it is if it turns out to be exact. To check this, we need to put it into the form $M(x, y)dx + N(x, y)dy = 0$. We find that

$$\begin{aligned}(2x^2y + 2x + 1)\frac{dy}{dx} &= -(2xy^2 + 2y) \\ (2x^2y + 2x + 1) dy &= -(2xy^2 + 2y) dx \\ (2xy^2 + 2y) dx + (2x^2y + 2x + 1) dy &= 0\end{aligned}$$

So $M(x, y) = 2xy^2 + 2y$ and $N(x, y) = 2x^2y + 2x + 1$. So $M_y = N_x = 4xy + 2$. So we look for a function $\psi(x, y)$ such that $\psi_x = M$ and $\psi_y = N$. Therefore

$$\begin{aligned}\psi(x, y) &= \int M(x, y) dx \\ &= \int (2xy^2 + 2y) dx \\ &= x^2y^2 + 2xy + h(y)\end{aligned}$$

To find $h(y)$ we use the condition that $\psi_y = N$. This tells us that $2xy^2 + 2x + h'(y) = 2x^2y + 2x + 1$. So $h'(y) = 1$ and therefore $h(y) = y$ (ignoring the arbitrary constant of integration). Solutions are given by setting $\psi(x, y)$ equal to a constant, so we have that our solution is given by

$$x^2y^2 + 2xy + y = c$$

(c) The numerator depends only on x and the denominator depends only on y so we can immediately conclude that it's a separable equation. Separating variables yields

$$\begin{aligned} \int e^y dy &= \int x^2 + 2x dx \\ e^y &= \frac{1}{3}x^3 + x^2 + C \end{aligned}$$

Problem 2. Solve each of the following initial value problems and state where the solutions are defined.

(a) This is a separable equation and so we have

$$\begin{aligned} \int (2y - 2) dy &= \int (2x - 4) dx \\ y^2 - 2y &= x^2 - 4x + C \end{aligned}$$

Using our initial value, we get $0 = C$ and so $y^2 - 2y = x^2 - 4x$. To figure out where the solution is defined, we put the solution into explicit form. Either use the quadratic formula or complete the square to obtain $y = 1 \pm \sqrt{x^2 - 4x + 1}$. Use the initial value to conclude that $y = 1 - \sqrt{x^2 - 4x + 1}$. This only makes sense when $x^2 - 4x + 1$ is nonnegative. Completing the square gives us that $x^2 - 4x + 1 = (x - 2)^2 - 3$, which is nonnegative for $x \geq 2 + \sqrt{3}$ and for $x \leq 2 - \sqrt{3}$. Our initial value is given for $x = 0$, which is in the second interval, and therefore our solution is only defined there. But also note that the original ODE is not defined for $y = 1$ and so we actually need the term $\sqrt{x^2 - 4x + 1}$ to be nonzero. This excludes the point $x = 2 - \sqrt{3}$ and so our solution is defined for $x < 2 - \sqrt{3}$.

(b) We form the characteristic equation $r^2 + r - 2 = 0$. This factors into $(r - 1)(r + 2) = 0$ and so the general solution is $y(t) = c_1 e^t + c_2 e^{-2t}$. Then $y'(t) = c_1 e^t - 2c_2 e^{-2t}$. Applying our initial conditions yields $y(0) = c_1 + c_2 = 0$ and $y'(0) = c_1 - 2c_2 = 1$. We can solve the two equations simultaneously for c_1 and c_2 to get that $c_1 = \frac{1}{3}$ and $c_2 = -\frac{1}{3}$. Therefore our solution is $y = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$. Our solution is defined for all real numbers t .

Problem 3.

(a) Computing the derivatives of y_1 we get $y_1' = 1$ and $y_1'' = 0$. Therefore $y_1'' - y_1' = 0 - 1 = -1$, as desired. Likewise, computing the derivatives of y_2 gives $y_2' = 1 + e^t$ and $y_2'' = e^t$ and so $y_2'' - y_2' = e^t - (1 + e^t) = -1$, again as desired.

(b) Following the hint, we compute $W(y_1, y_2) = y_1 y_2' - y_2 y_1' = t(1 + e^t) - (t + e^t) = (t - 1)e^t$. Note then that the Wronskian is equal to 0 at $t = 1$ and is nonzero everywhere else. However, it is a consequence of Theorem 3.3.3 (see page 156 of the book) that solutions to $y'' + p(t)y' + q(t)y = 0$ in an interval I have the property that either their Wronskian is always zero in I or their Wronskian is never zero in I . y_1 and y_2 do not have this property for intervals containing $t = 1$ and so they cannot be solutions to any such ODE on an interval containing $t = 1$. Note that we have not contradicted our result in part (a) because the right hand side of the ODE in part (a) is nonzero.

Problem 4. Determine whether the following pairs of functions are linearly independent or linearly dependent over the given intervals.

(a) These two functions are linearly independent. One way to show this is to compute the Wronskian and show that there are points where it is nonzero.

(b) These two functions are linearly dependent. Using the angle sum formula we obtain $\sin(t + \pi) = \sin t \cos \pi + \sin \pi \cos t = -\sin t$. Therefore $y_1 + y_2 = \sin t - \sin t = 0$ which implies that the two functions are linearly dependent.

Problem 5

(a) We use Newton's second law $F = m \frac{dv}{dt}$ where F is the net force acting on the object. The forces acting on the ball are gravity and air resistance. We end up with

$$\begin{aligned} 2\frac{dv}{dt} &= 20 - 4v \\ \frac{dv}{dt} &= 10 - 2v \end{aligned}$$

Since we are dropping the ball, the initial velocity is zero, so our initial condition is $v(0) = 0$. This equation is both linear and separable. Either way we choose to solve it we will obtain a solution of $v(t) = 5 + Ae^{-2t}$. After applying our initial value, we find that $v(t) = 5 - 5e^{-2t}$.

(b) Since we are being asked about the relationship between position x and time t , we need to find an equation that relates x and t . We can use the fact that $v = \frac{dx}{dt}$ to write our solution in part (a) as a differential equation for $x(t)$. So we have $\frac{dx}{dt} = 5 - 5e^{-2t}$. If we take the convention that the top of the building is at height $x = 0$ (so that the ground is at height $x = 100$), our initial condition becomes $x(0) = 0$. Note that the right hand side of our differential equation depends only on t and so we can solve it simply by integrating. We then obtain

$$x(t) = 5t + \frac{5}{2}e^{-2t} + C$$

Applying our initial condition we obtain $x(0) = \frac{5}{2} + C = 0$, so $C = -\frac{5}{2}$. So $x(t) = 5t + \frac{5}{2}e^{-2t} - \frac{5}{2}$. We want to find when the ball hits the ground, i.e., we want t_0 such that $x(t_0) = 100$. The problem then reduces to finding the solution of $100 = 5t + \frac{5}{2}e^{-2t} - \frac{5}{2}$. We cannot solve this exactly, but an approximate solution is $t = 20.5s$. (Note that on the exam you aren't allowed to use calculators, so a problem that is exactly like this is won't show up, but you may be asked some other question that requires you to find the function $x(t)$).

(c) The velocity will be constant if the acceleration $\frac{dv}{dt}$ is zero. From the differential equation $\frac{dv}{dt} = 10 - 2v$ we see that this occurs when $v_0 = 5m/s$.