

Math 319 Solutions to Review Problems for Exam 2

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Problem 1.

(a) Solving the characteristic equation gives a repeated root of $r_1 = r_2 = 2$, so the general solution is $y(t) = c_1 e^{2t} + c_2 t e^{2t}$ (or we may write it as $y(t) = k_1 e^{2(t-2)} + k_2 (t-2) e^{2(t-2)}$ to make the math easier when solving for the constants). In either case, we can apply our initial values to find that $y = (t-2)e^{2(t-2)} = -2e^{2t-4} + te^{2t-4}$.

(b) Solving the characteristic equation gives roots of $r_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $r_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. So the general solution is

$$y(t) = e^{-t/2} \left[c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right) \right]$$

Our initial conditions give $y(0) = c_1 = 2$ and $y'(0) = \frac{\sqrt{3}}{2}c_2 - \frac{1}{2}c_1 = 1$, so $c_1 = 2$ and $c_2 = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$. So our final answer is

$$y(t) = e^{-t/2} \left[2 \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{4\sqrt{3}}{3} \sin\left(\frac{\sqrt{3}}{2}t\right) \right]$$

Problem 2.

(a) $y_1(x) = x$, $y_1'(x) = 1$ and $y_1''(x) = 0$. So $x^2 y_1'' - x y_1' + y_1 = x^2(0) - x(1) + x = 0$, so y_1 is a solution.

(b) We assume a solution in the form $y(x) = y_1(x)v(x) = xv(x)$ and plug into the differential equation to solve for $v(x)$. Since $y = xv(x)$ we have that

$y' = xv'(x) + v(x)$ and $y'' = xv''(x) + 2v'(x)$. Then we have that (after some simplification) $x^2y'' - xy' + y = x^3v'' + x^2v'$. But we are assuming that y is a solution of the ODE, so we must have that $x^2y'' - xy' + y = 0$ and therefore $x^3v'' + x^2v' = 0$. Now if we let $w = v'$ then we are left with $x^3w' + x^2w = 0$, a first order linear ODE for w . In standard form this is $w' + \frac{1}{x}w = 0$. A routine calculation yields that the integrating factor is $\mu(x) = x$, and so we are left with solving $[xw]' = 0$. Then $xw = c_1$ and so $w = \frac{c_1}{x}$. But $w = v'$, so $v' = \frac{c_1}{x}$ and so $v = \int v' dx = \int \frac{c_1}{x} dx = c_1 \ln x + c_2$. Then solutions to the ODE are given by $y(x) = y_1(x)v(x) = xv(x) = c_1x \ln x + c_2x$. This is a solution for any choice of c_1 and c_2 , so for instance choosing $c_1 = 1$ and $c_2 = 0$ gives a solution of $y_2(x) = x \ln x$.

Problem 3.

(a) The characteristic equation has a repeated root $r_1 = r_2 = 1$, so the solution of the homogeneous problem is $y_c(t) = c_1e^t + c_2te^t$. We can find a particular solution using undetermined coefficients. Since the right hand side is $2e^t$ we would like a solution to have the form of Ae^t . But Ae^t solves the homogeneous problem. We have the same issue with Ate^t , so we look for a particular solution of the form $Y(t) = At^2e^t$. Plugging into the differential equation we find that $A = 1$, so the general solution is $y(t) = c_1e^t + c_2te^t + t^2e^t$.

(b) The characteristic equation has complex roots of $r_1 = 2i$ and $r_2 = -2i$, so the solution of the homogeneous problem is $y_c(t) = c_1 \cos(2t) + c_2 \sin(2t)$. For the problem $y'' + 4y = 8t^2$, we guess a solution of the form $Y_1(t) = At^2 + Bt + C$ (the right hand side is a second degree polynomial, so we guess a second degree polynomial solution). We find that $Y_1'' + 4Y_1 = 4At^2 + 4Bt + (2A + 4C)$ and we want $Y_1'' + 4Y_1 = 8t^2$, so we obtain $A = 2, B = 0, C = -1$. For the problem $y'' + 4y = 10e^{-2t}$ we guess a solution of the form $Y_2(t) = De^{-2t}$. Plugging into the differential equation we find that $D = \frac{5}{4}$. Our general solution is $y(t) = y_c(t) + Y_1(t) + Y_2(t)$ so we obtain a solution of

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t) + 2t^2 - 1 + \frac{5}{4}e^{-2t}$$

(c) Note that since the right hand side is not a sum of products of polynomials, exponentials, sines and cosines, we cannot use the method of undetermined coefficients, so we will instead use variation of parameters. The roots of the characteristic equation are $r_1 = i$ and $r_2 = -i$, so we can write our two linearly independent solutions of the homogeneous problem as $y_1(t) = \cos t$

and $y_2(t) = \sin(t)$. We can compute that the Wronskian of y_1 and y_2 is $W(y_1, y_2)(t) = \cos^2 t + \sin^2 t = 1$. The ODE is already in standard form, so we have that $g(t) = \sec t$. Then a particular solution is given by

$$\begin{aligned} Y(t) &= -y_1(t) \int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} dt + y_2(t) \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} dt \\ &= -\cos t \int \sin t \sec t dt + \sin t \int \cos t \sec t dt \\ &= -\cos t \int \frac{\sin t}{\cos t} dt + \sin t \int dt \\ &= \cos t \ln(\cos t) + t \sin t \end{aligned}$$

We compute the first integral by making the substitution $u = \cos t$. Also note that we are allowed to write $\ln(\cos t)$ instead of $\ln|\cos t|$ because we are restricted to the interval $0 < t < \pi/2$, on which cosine is always positive. Our general solution is given by $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$, so we have

$$y(t) = c_1 \cos t + c_2 \sin t + \cos t \ln(\cos t) + t \sin t$$

Problem 4.

(a) Compute the derivatives of y_1 and plug into the ODE, as in Problem 2(a). Then do the same for y_2 .

(b) We begin by putting the equation in standard form $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 1$. Therefore $g(x) = 1$. We compute that the Wronskian of y_1 and y_2 is $W(y_1, y_2)(x) = 2x^2 - x^2 = x^2$. So a particular solution is given by

$$\begin{aligned} Y(x) &= -y_1(x) \int \frac{y_2(x)g(x)}{W(y_1, y_2)(x)} dx + y_2(x) \int \frac{y_1(x)g(x)}{W(y_1, y_2)(x)} dx \\ &= -x \int dx + x^2 \int \frac{1}{x} dx \\ &= -x^2 + x^2 \ln x \end{aligned}$$

Since we restrict our domain to $x > 0$ we are allowed to write $\ln x$ instead of $\ln|x|$. The general solution is given by $y(x) = c_1 y_1(x) + c_2 y_2(x) + Y(x)$, so we have

$$y(x) = c_1x + c_2x^2 + x^2 \ln x$$

(Note that we have absorbed the term of $-x^2$ that appeared in $Y(x)$ into the arbitrary constant c_2 .)

Problem 5.

The Laplace transform is defined by

$$F(s) = \int_{t=0}^{t=\infty} f(t)e^{-st} dt = \lim_{A \rightarrow \infty} \int_{t=0}^{t=A} f(t)e^{-st} dt$$

wherever this integral converges. Thus we are left with finding

$$\begin{aligned} F(s) &= \lim_{A \rightarrow \infty} \int_{t=0}^{t=A} \frac{e^{at} + e^{-at}}{2} e^{-st} dt \\ &= \lim_{A \rightarrow \infty} \frac{1}{2} \int_{t=0}^{t=A} [e^{(a-s)t} + e^{-(a+s)t}] dt \\ &= \frac{1}{2} \lim_{A \rightarrow \infty} \left[\frac{e^{(a-s)t}}{a-s} - \frac{e^{-(a+s)t}}{a+s} \right]_{t=0}^{t=A} \end{aligned}$$

One can check that this limit will exist (and hence the integral will converge) if and only if $s > |a|$, in which case we will have

$$F(s) = \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right] = \frac{s}{s^2 - a^2}$$

Problem 6.

We need to express $F(s)$ so that it is a linear combination of terms whose inverse Laplace transforms we can look up in a table. Using a partial fraction decomposition to turn $\frac{3s+5}{s(s^2+2s+5)}$ into $\frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$ we find that $A = 1$, $B = -1$ and $C = 1$. The term $s^2 + 2s + 5$ is an irreducible quadratic, so we complete the square to rewrite it as $(s + 1)^2 + 4$. Thus we have that

$$F(s) = \frac{1}{s} + \frac{-s + 1}{(s + 1)^2 + 4}$$

Noting that $-s + 1 = -s - 1 + 2 = -(s + 1) + 2$, we have

$$F(s) = \frac{1}{s} - \frac{s+1}{(s+1)^2+4} + \frac{2}{(s+1)^2+4}$$

We can now use a table of Laplace transforms to conclude that

$$f(t) = 1 - e^{-t} \cos(2t) + e^{-t} \sin(2t)$$

Problem 7.

Taking Laplace transforms of both sides yields

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s+1} + \frac{1}{s^2}$$

where $Y(s)$ is the Laplace transform of $y(t)$. Applying our initial values and solving for $Y(s)$ gives us that

$$Y(s) = \frac{2}{(s+1)(s^2+1)} + \frac{1}{s^2(s^2+1)} + \frac{1}{s^2+1}$$

The term $\frac{1}{s^2+1}$ is already in a form where we can look up its inverse Laplace transform, but we need to perform partial fraction decompositions on the other two terms. For the first term we get (after some algebra)

$$\frac{2}{(s+1)(s^2+1)} = \frac{1}{s+1} + \frac{-s+1}{s^2+1} = \frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

For the second term we get (again after some algebra)

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

Putting this all together, we obtain

$$Y(s) = \frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s^2}$$

We then find the inverse Laplace transform using a table, getting a final answer of

$$y(t) = e^{-t} - \cos t + \sin t + t$$

Problem 8

Our general equation is $mu'' + \gamma u + ku = F(t)$, where m is the mass, γ is the coefficient of our damping force, k is the spring constant and $F(t)$ is the external force. The mass is given by $m = \frac{w}{g}$ where w is the weight and g is the acceleration of gravity (which is 32 feet per second squared). So $m = \frac{4}{32} = \frac{1}{8}$ (given in pounds times seconds squared per foot). The spring constant is given by $k = \frac{w}{L}$ where L is the amount by which the mass stretches the spring. So (converting L to feet so as to be consistent with our units) we have $k = \frac{4}{8/12} = 6$ (given in pounds per foot). There is no damping or driving force, so $\gamma = 0$ and $F(t) = 0$. So, converting our initial conditions to ft and ft/s, we have an initial value problem of

$$\frac{1}{8}u'' + 6u = 0, u(0) = \frac{1}{2}, u'(0) = 0$$

Problem 9

We are given that $m = 3$ kg (note that kilograms are a unit of mass, not weight). The weight is given by $w = mg$ where $g = 9/8$ meters per second squared is the acceleration of gravity. Therefore (converting L to meters so as to be consistent with our units) we have that $k = \frac{(3)(9.8)}{.15} = 196$ Newtons per meter. To find γ we use the fact that the magnitude of the viscous force is equal to $\gamma u'$ and so $\gamma = \frac{3}{3} = 1$ Ns/m. We are given that $F(t) = 5 \cos(2t) + 2 \sin(2t)$ N, so our initial value problem is

$$3u'' + u' + 196u = 5 \cos(2t) + 2 \sin(2t), u(0) = 0, u'(0) = 2$$