Math 319 Solutions to Review Problems for Final Exam

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Problem 1.

(a) This is a separable differential equation. Separating varibles gives us

$$\int y\,dy = \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$$

To compute the integral on the right, make the substitution $u = \sqrt{x}$ so that $du = \frac{1}{2\sqrt{x}}\,dx$. Using this, we find that solutions are given implicitly by

$$\frac{1}{2}y^2 = 2\sin\sqrt{x} + C$$

(b) This is a linear differential equation. Written in standard form, it becomes

$$y' + \frac{1}{x}y = e^{x^2}$$

So we have an integrating factor of

$$\mu(x) = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$$

Thus the problem is now to solve $[xy]' = xe^{x^2}$. We do this by integrating both sides.
Note that to integrate the right hand side, we need to make the substitution $u = x^2$ so that $du = 2\,dx$.
After integrating both sides we will obtain $xy = \frac{1}{2}e^{x^2} + C$. In explicit form this becomes

$$y = \frac{1}{2x} e^{x^2} + \frac{C}{x}$$

(c) The given equation is neither separable nor linear, so our only hope of solving it is for it to be exact. Rewriting in differential form we obtain $(y^2 + 2xy)\,dx + (x^2 + 2xy - 1)\,dy = 0$. So letting $M(x, y) = y^2 + 2xy$ and $N(x, y) = x^2 + 2xy - 1$ we see that $M_y = N_x = 2x + 2y$ so indeed the equation is exact. Then we need to find $\psi(x, y)$ so that $\psi_x = M$ and $\psi_y = N$. Integrating M with respect to x gives us

$$\psi(x, y) = \int M(x, y)\,dx = \int (y^2 + 2xy)\,dx = xy^2 + x^2 y + g(y)$$

where $g(y)$ is an arbitrary function of y. Directly computing the partial derivative with respect to y gives $\psi_y = 2xy + x^2 + g'(y)$. But we also know that $\psi_y = N(x, y) = x^2 + 2xy - 1$. Comparing these two expressions gives us $g'(y) = -1$, so $g(y) = -y$ (ignoring the arbitrary constant of integration for the time being) and so $\psi(x, y) = xy^2 + x^2 y - y$. Solutions are given implicitly by setting ψ equal to an arbitrary constant, so our solution is given by

$$xy^2 + x^2 y - y = c$$

Don't forget this last step! On the exam, if you just find ψ and forget to set it equal to a constant to find solutions, you will lose points.

Problem 2.

This is a separable differential equation, so solutions are found by integrating both sides of $2y\,dy = (2x - 3)\,dx$. So we have a solution of $y^2 = x^2 - 3x + C$. Applying our initial condition gives us that $C = -1$, so $y^2 = x^2 - 3x - 1$. Taking square roots of both sides gives us $y = \pm\sqrt{x^2 - 3x - 1}$. Our initial condition is $y(4) = -\sqrt{3}$, so we need to take the negative square root. Therefore our solution is

$$y = -\sqrt{x^2 - 3x - 1}$$

Clearly, this solution will only be valid for $x^2 - 3x - 1 \geq 0$. Completing the square gives us that our condition is

$$\left(x - \frac{3}{2}\right)^2 - \frac{13}{4} \geq 0$$

Rearranging this we have

$$\left(x - \frac{3}{2}\right)^2 \geq \frac{13}{4}$$

Taking square roots gives us that either $x - \frac{3}{2} \geq \frac{\sqrt{13}}{3}$ or $x - \frac{3}{2} \leq -\frac{\sqrt{13}}{2}$. So our possibilities are $x \geq \frac{\sqrt{13}+3}{2}$ or $x \leq \frac{3-\sqrt{13}}{2}$. Now our initial condition is given at $x = 4$ and $4 > \frac{\sqrt{13}+3}{2}$ (we can see this by noting that $\frac{\sqrt{13}+3}{2} < \frac{\sqrt{16}+3}{2} = \frac{7}{2} < 4$) so we want to keep the first interval. So we need $x \geq \frac{\sqrt{13}+3}{2}$. But at $x = \frac{\sqrt{13}+3}{2}$, we have that $y = 0$, and so at that point we would be dividing by 0 in our differential equation $\frac{dy}{dx} = \frac{2x-3}{4}$, so we also exclude that point. So our solution is valid for

$$x > \frac{\sqrt{13}+3}{2}$$

Problem 3.

(a) Solving the characteristic equation gives us roots of $r_1 = 1$ and $r_2 = -3$ so our general solution is $y = c_1 e^t + c_2 e^{-3t}$. Applying our initial conditions gives $y(0) = c_1 + c_2 = 4$ and $y'(0) = c_1 - 3c_2 = 0$. Solving this system gives us $c_1 = 3$ and $c_2 = 1$ and so we have a solution of

$$y(t) = 3e^t + e^{-3t}$$

(b) The characteristic equation has a repeated root of $r = -\frac{1}{3}$ so the general solution is $y(t) = c_1 e^{-t/3} + c_2 t e^{-t/3}$. Applying the initial conditions gives $y(0) = c_1 = 6$ and $y'(0) = -\frac{1}{3}c_1 + c_2 = 1$. This will give us a solution of

$$y(t) = 6e^{-t/3} + 3te^{-t/3}$$

(c) The characteristic equation has complex roots of $r = -1 \pm 2i$ which gives us a general solution of $y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$. Our initial conditions then give us that $y(0) = c_1 = 1$ and $y'(0) = -c_1 + 2c_2 = 1$ which gives us a solution of

$$y(t) = e^{-t}\cos(2t) + e^{-t}\sin(2t)$$

Problem 4.

(a) We will use the method of undetermined coefficients. The solution to the associated homogeneous problem is $y_c(t) = c_1e^{2t} + c_2e^{-t}$.
The right hand side is of the form of a first degree polynomial multiplied by an exponential, so matching that form would give us a particular solution of the form $Y(t) = (At + B)e^{2t}$. But the term Be^{2t} solves the homogeneous problem, so we need to multiply through by t, giving us a particular solution of the form $Y(t) = t(At + B)e^{2t} = At^2e^{2t} + Bte^{2t}$.
Taking derivatives and simplifying, we find that $Y'' - Y' - 2Y = 6Ate^{2t} + (2A + 3B)e^{2t}$. Therefore we have that $A = \frac{1}{6}$ and $B = -\frac{1}{9}$. We then form our general solution by taking $y(t) = y_c(t) + Y(t)$, giving us

$$y(t) = c_1e^{2t} + c_2e^{-t} + \frac{1}{6}t^2e^{2t} - \frac{1}{9}te^{2t}$$

Note that we found the form of our particular solution in a systematic manner, rather than by guess and check. This is how we should always approach undetermined coefficient problems. Guess and check is an inefficient way to solve problems such as these, especially in an exam situation where there is little time to waste. See Table 3.6.1 (page 181 of the text) for a listing of the forms of the particular solution needed for each case.

(b) We will again use the method of undetermined coefficients. The solution of the associated homogeneous equation is $y_c(t) = c_1e^t \cos t + c_2e^t \sin t$.
Since our nonhomogeneous term is a sum, we split into two problems

$$y'' - 2y' + 2y = \cos t \quad (1)$$
$$y'' - 2y' + 2y = 4t \quad (2)$$

For equation (1), the right hand side is $\cos t$, which gives a form of $Y_1(t) = A\cos t + B\sin t$ for the particular solution. Neither term solves the homogeneous equation (since there's no e^t multiplying the cosine or the sine), so this is the proper form of the solution.
Taking derivatives and simplifying, we have that $Y_1'' - 2Y_1' + 2Y_1 = (A - 2B)\cos t + (2A + B)\sin t$. Therefore we have that $A - 2B = 1$ and $2A + B = 0$. Solving these simultaneously gives us $A = \frac{1}{5}$ and $B = -\frac{2}{5}$ and therefore $Y_1(t) = \frac{1}{5}\cos t - \frac{2}{5}\sin t$.

For equation (2), the right hand side is a first degree polynomial, which gives a form of $Y_2(t) = Ct + D$ for the particular solution. Neither term solves the homogenous equation, so this is the proper form of our particular solution.

Taking derivatives and simplifying, we have that $Y_2'' - 2Y_2' + 2Y_2 = 2Ct + (2D - 2C)$ and so $C = 2$ and $D = 2$. So $Y_2(t) = 2t + 2$. Our general solution is $y(t) = y_c(t) + Y_1(t) + Y_2(t)$ and so we have

$$y(t) = c_1 e^t \cos t + c_2 e^t \sin t + \frac{1}{5}\cos t - \frac{2}{5}\sin t + 2t + 2$$

(c) The right hand side is not of the form of a sum of products of polynomials, exponentials, sines and cosines, so we cannot use undetermined coefficients. We instead proceed by variation of parameters. The solution to the associated homogeneous problem is $y(t) = c_1 e^t + c_2 t e^t$, so our two linearly independent solutions are $y_1(t) = e^t$ and $y_2(t) = te^t$. Our equation is already in standard form, so we have $g(t) = \frac{e^t}{t^2}$. Finally, by direct computation, we see that the Wronskian of y_1 and y_2 is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = (t+1)e^{2t} - te^{2t} = e^{2t}$$

We know that a particular solution is given by

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\,dt$$

Plugging in $y_1(t) = e^t$, $y_2(t) = te^t$, $g(t) = \frac{e^t}{t^2}$ and $W(y_1, y_2)(t) = e^{2t}$ and simplifying, we get

$$Y(t) = -e^t \int \frac{1}{t}\,dt + te^t \int \frac{1}{t^2}\,dt = -e^t \ln t - e^t$$

Therefore the general solution is

$$y(t) = c_1 e^t + c_2 t e^t - e^t \ln t$$

Note that we absorbed the term of $-e^t$ from our particular solution into the arbitrary constant c_1.

Problem 5.

(a) By calculating the derivatives of y_1 and y_2 and substituting them into the differential equation, we find that they are indeed solutions.

(b) In standard form, our equation becomes $y'' + \frac{4}{t}y' + \frac{2}{t^2}y = \frac{e^t}{t^2}$, so we have that $g(t) = \frac{e^t}{t^2}$ (Don't forget to the put equation in standard form! If you forget, you'll get the wrong form for g). From part (a) we have $y_1(t) = \frac{1}{t^2}$ and $y_2(t) = \frac{1}{t}$. Then we find that

$$W(y_1, y_2)(t) = \begin{vmatrix} \frac{1}{t^2} & \frac{1}{t} \\ -\frac{2}{t^3} & -\frac{1}{t^2} \end{vmatrix} = -\frac{1}{t^4} + \frac{2}{t^4} = \frac{1}{t^4}$$

We know that a particular solution is given by

$$Y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\, dt + y_2(t) \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\, dt$$

Plugging in our values of y_1, y_2, g and W and simplifying, we obtain

$$Y(t) = -\frac{1}{t^2} \int te^t\, dt + \frac{1}{t} \int e^t\, dt$$

To compute the first integral, use integration by parts with $u = t$ and $dv = e^t\, dt$. After computing each integral, we obtain

$$Y(t) = -\frac{1}{t^2} \left[te^t - e^t\right] + \frac{1}{t} e^t = \frac{e^t}{t^2}$$

Therefore, our general solution is

$$y(t) = c_1 \frac{1}{t^2} + c_2 \frac{1}{t} + \frac{e^t}{t^2}$$

Problem 6.

(a) By calculating the derivatives of y_1 and substituting them into the differential equation, we find that it is indeed a solution.

(b) We assume a solution of the form $y(t) = y_1(t)v(t)$, where v is an as yet unspecified function of t. Therefore, we have that

$$\begin{aligned} y(t) &= \frac{1}{t^2} v(t) \\ y'(t) &= -\frac{2}{t^3} v(t) + \frac{1}{t^2} v'(t) \\ y''(t) &= \frac{6}{t^4} v(t) - \frac{4}{t^3} v'(t) + \frac{1}{t^2} v''(t) \end{aligned}$$

Since y is assumed to be a solution to $t^2 y'' + 2ty' - 2y = 0$, we can substitute the y, y' and y'' we calculated above into the ODE. When we do this, after some simplification we will obtain $v'' - \frac{2}{t}v' = 0$. This is a second order ODE for v, but if we let $w = v'$ we obtain $w' - \frac{2}{t}w = 0$, a first order ODE for w. This ODE is both linear and separable. Using either method, we obtain as a solution $w = Ct^2$. Since $w = v'$, we have that $v = \int w\,dt = \int Ct^2\,dt = c_1t^3 + c_2$ (we have absorbed the term of $\frac{1}{3}$ into the arbitrary constant c_1). Our solution is then

$$y(t) = \frac{1}{t^2}v(t) = \frac{1}{t^2}[c_1t^3 + c_2] = c_1t + c_2\frac{1}{t^2}$$

This is a solution for any choice of c_1 and c_2, and so long as we let c_1 be nonzero our solution will be linearly independent from $y_1(t)$. So, for instance, if we choose $c_1 = 1$ and $c_2 = 0$ we obtain a solution of

$$y_2(t) = t$$

Problem 7.

(a) Taking the Laplace transform of both sides gives us $s^2Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] + 3Y(s) = \frac{3}{s}$ where $Y(s)$ is the Laplace transform of the solution y. After plugging in our initial conditions and solving for Y, we obtain

$$Y(s) = \frac{3}{s(s^2 - 2s + 3)}$$

After a partial fraction decomposition we obtain

$$Y(s) = \frac{1}{s} + \frac{-s + 2}{s^2 - 2s + 3}$$

The denominator of the second term doesn't factor, but if we complete the square we obtain $s^2 - 2s + 3 = (s-1)^2 + 2$. Splitting up our term so we can match rules 9 and 10 from the table on page 319 gives us

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s-1}{(s-1)^2 + 2} + \frac{1}{(s-1)^2 + 2} \\ &= \frac{1}{s} - \frac{s-1}{(s-1)^2 + 2} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s-1)^2 + 2} \end{aligned}$$

Taking inverse Laplace transforms gives us

$$y(t) = 1 + \frac{1}{\sqrt{2}} e^t \sin(\sqrt{2}t) - e^t \cos(\sqrt{2}t)$$

(b) Taking the Laplace transform of both sides gives us

$$s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0) - [s^2 Y(s) - sy(0) - y'(0)] = \frac{1}{s-1}$$

Applying our initial conditions and solving for $Y(s)$ gives us

$$\begin{aligned} Y(s) &= \frac{1}{s^2(s-1)^2} + \frac{1}{s^2(s-1)} \\ &= \frac{s}{s^2(s-1)^2} \end{aligned}$$

We perform a partial fraction decomposition by writing

$$\frac{s}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{D}{(s-1)^2}$$

Solving for the constants A, B, C and D gives us

$$Y(s) = \frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2}$$

Taking inverse Laplace transforms gives

$$y(t) = 1 - e^t + te^t$$

Problem 8

(a) By taking the determinant of $A - \lambda I$, we see that we have eigenvalues of $\lambda_1 = 4$ and $\lambda_2 = -1$. Corresponding to these eigenvalues are the eigenvectors

$$\xi^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So our general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

Note that we could have made different arbitrary choices when solving for our eigenvectors and obtained somewhat different looking (but equivalent) answers.

(b) We get a repeated eigenvalue of $\lambda = 2$ and associated with that eigenvalue is the eigenvector

$$\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We need to find a generalized eigenvector η. This will be goverened by the equation $(A - \lambda I)\eta = \xi$, which gives us $\eta_1 + \eta_2 = -1$. If we let $\eta_1 = 0$ we obtain $\eta_2 = -1$ and so we obtain a solution of

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} \right]$$

Note that, since there were two instances where we made arbitary choices, there are many different ways to write the correct answer. For instance, if we had instead chosen the eigenvector

$$\xi = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Then our equation governing η would instead be $\eta_1 + \eta_2 = 1$, and we could have then obtained the following answer (which is equivalent to the answer we found above)

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} \right]$$

(c) We obtain eigenvalues of $\lambda = 2 \pm i$. Working with the eigenvalue $\lambda = 2+1$, we get two equations governing the associated eigenvector ξ. One of these equations is $(1-i)\xi_1 + 2\xi_2 = 0$ and the other is $-\xi_1 + (-1-i)\xi_2 = 0$. We know from experience that the two equations governing an eigenvector of a 2×2 matrix are equivalent, so we can work with either equation. For instance, working with the second equation and letting $\xi_2 = 1$, we obtain $\xi_1 = -1 - i$. So we have a solution of

$$\begin{pmatrix} -1-i \\ 1 \end{pmatrix} e^{(2+i)t}$$

Applying Euler's identity to $e^{(2+i)t} = e^{2t}e^{it}$, we see that our solution is

$$\begin{pmatrix} -1-i \\ 1 \end{pmatrix} e^{2t}[\cos t + i \sin t]$$

Multiplying the term of $\cos t + i \sin t$ through and splitting into real and imaginary parts, we obtain

$$\begin{pmatrix} -\cos t + \sin t \\ \cos t \end{pmatrix} e^{2t} + i \begin{pmatrix} -\cos t - \sin t \\ \sin t \end{pmatrix} e^{2t}$$

So our real part $\mathbf{u}$ and imaginary part $\mathbf{v}$ are given by

$$\mathbf{u} = \begin{pmatrix} -\cos t + \sin t \\ \cos t \end{pmatrix} e^{2t}$$

$$\mathbf{v} = \begin{pmatrix} -\cos t - \sin t \\ \sin t \end{pmatrix} e^{2t}$$

Our general (real-valued) solution is $\mathbf{x}(t) = c_1\mathbf{u} + c_2\mathbf{v}$ and so we have a general solution of

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -\cos t + \sin t \\ \cos t \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -\cos t - \sin t \\ \sin t \end{pmatrix} e^{2t}$$

Again keep in mind that there are many equivalent ways to write the correct answer (some of which may look very different at a glance) depending on which eigenvector we choose. For instance, if we had chosen $\xi_1 = 2$ we would get from the equation $(1-i)\xi_1 + 2\xi_2 = 0$ that $\xi_2 = -1 + i$ and so our eigenvector would be

$$\xi = \begin{pmatrix} 2 \\ -1+i \end{pmatrix}$$

From this we would find a general solution (equivalent to the one we found above) of

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2\cos t \\ -\cos t - \sin t \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 2\sin t \\ \cos t - \sin t \end{pmatrix} e^{2t}$$

Problem 9 Consider the initial value problem $y'' - y = 0$, $y(0) = 2$, $y'(0) = 0$

(a) The characteristic equation is $r^2 - 1 = 0$, so our general solution is $y(t) = c_1 e^t + c_2 e^{-t}$. Applying our initial values, we find that $c_1 = 1$ and $c_2 = 1$, so we have $y(t) = e^t + e^{-t}$.

(b) Letting $x_1 = y$ and $x_2 = y'$, we immediately get that $x_1' = x_2$. Moreover, from the equation $y'' - y = 0$, we have $x_2' - x_1 = 0$, i.e. $x_2' = x_1$. Our initial conditions become $x_1(0) = 2$ and $x_2(0) = 0$. So our system of equations is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_1 \\ x_1(0) &= 2 \\ x_2(0) &= 0 \end{aligned}$$

(c) We can rewrite our system in matrix form as

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

where we have written

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The eigenvalues of our matrix are $\lambda_1 = 1$ and $\lambda_2 = -1$ and we can associate to each the eigenvectors

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Therefore our general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

Applying our initial condition gives us

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

which tells us that $c_1 + c_2 = 2$ and $c_1 - c_2 = 0$. So $c_1 = 1$ and $c_2 = 1$, which gives us that

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

Note that the first component tells us that $x_1(t) = e^t + e^{-t}$ and the second component tells us that $x_2(t) = e^t - e^{-t}$. But $x_1 = y$ and $x_2 = y'$, so what this really tells us is that $y(t) = e^t + e^{-t}$ and $y'(t) = e^t - e^{-t}$, which agrees with our answer from part (a).

Problem 10. We have that $\mathbf{\Phi}(t) = \begin{pmatrix} \mathbf{x}^{(1)}(t) & \mathbf{x}^{(2)}(t) \end{pmatrix}$ where $\mathbf{x}^{(1)}$ solves the initial value problem

$$\mathbf{x}' = A\mathbf{x}, \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and $\mathbf{x}^{(2)}$ solves the initial value problem

$$\mathbf{x}' = A\mathbf{x}, \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$. Associated to these eigenvalues are the eigenvectors

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \xi^{(1)} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

So the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{2t}$$

To find $\mathbf{x}^{(1)}$ we set

$$\mathbf{x}^{(1)}(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The first component tells us that $c_1 + 3c_2 = 1$ and the second component tells us that $-c_1 - 2c_2 = 0$. This gives us $c_1 = -2$ and $c_2 = 1$ and so we have that

$$\mathbf{x}^{(1)}(t) = -2\begin{pmatrix}1\\-1\end{pmatrix}e^t + \begin{pmatrix}3\\-2\end{pmatrix}e^{-t} = \begin{pmatrix}-2e^t + 3e^{-t}\\2e^t - 2e^{-t}\end{pmatrix}$$

We can similarily find that

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix}-3e^t + 3e^{-t}\\3e^t - 2e^{-t}\end{pmatrix}$$

Putting these answers together, we find that

$$\mathbf{\Phi}(t) = \begin{pmatrix}-2e^t + 3e^{-t} & -3e^t + 3e^{-t}\\2e^t - 2e^{-t} & 3e^t - 2e^{-t}\end{pmatrix}$$

Note that there are many other possible ways to find $\mathbf{\Phi}(t)$, such as the method outlined in the text using the matrix $\exp(At)$.

Problem 11.

(a) Solutions to the system $\mathbf{x}' = A\mathbf{x}$ are given by $\mathbf{x} = \xi e^{\lambda t}$ where λ is an eigenvalue of A and ξ is the associated eigenvector. Therefore the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -1$ and the associated eigenvectors are

$$\xi^{(1)} = \begin{pmatrix}2\\3\end{pmatrix}, \qquad \xi^{(2)} = \begin{pmatrix}1\\2\end{pmatrix}$$

(b) We use the fact $A = T\Lambda T^{-1}$, where

$$\Lambda = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}, \qquad T = \begin{pmatrix}\xi^{(1)} & \xi^{(2)}\end{pmatrix}$$

So in our case we have

$$\Lambda = \begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}, \qquad T = \begin{pmatrix}2 & 1\\3 & 2\end{pmatrix}$$

We can readily compute that the inverse of T is

$$T^{-1} = \begin{pmatrix}2 & -1\\-3 & 2\end{pmatrix}$$

Therefore we have that

$$A = T\Lambda T^{-1} = \begin{pmatrix}2 & 1\\3 & 2\end{pmatrix}\begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}2 & -1\\-3 & 2\end{pmatrix}$$

Multiplying this out, we see that

$$A = \begin{pmatrix} 11 & -6 \\ 18 & -10 \end{pmatrix}$$