An Example of the Laplace Transform

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1 The Problem

Let $f(t) = t$, find the Laplace transform $F(s)$ of f.

2 The Solution

The Laplace transform of f is defined as $F(s) = \int_0^\infty f(t)e^{-st}\,dt$, wherever this converges. Therefore we need to find $F(s) = \int_0^\infty te^{-st}\,dt$, wherever it converges.

A few notes before we begin. First, note that we are integrating over the variable t and therefore **we regard s as a constant throughout the integration**. To emphasize the fact that we are integrating over t and not s, I will write

$$F(s) = \int_{t=0}^{t=\infty} te^{-st}\,dt$$

Second, note that if $s < 0$ then $-st > 0$ and so $\int_{t=0}^\infty te^{-st}\,dt$ will diverge. If $s = 0$ we are left with $\int_{t=0}^\infty t\,dt$, which also diverges. Therefore, the Laplace transform of f will only be defined if $s > 0$. Now, to actually calculate $F(s)$ we will use integration by parts with $u = t$ and $dv = e^{-st}dt$. Then $du = dt$ and $v = -\frac{1}{s}e^{-st}$ (remember, we treat s as though it is a constant throughout the integration). Thus we are left with

$$\begin{aligned}
F(s) &= \int_{t=0}^{t=\infty} te^{-st}\,dt \\
&= \lim_{A\to\infty} \int_{t=0}^{t=A} te^{-st}\,dt \\
&= \lim_{A\to\infty} \left[-\frac{t}{s}e^{-st}\Big|_{t=0}^{t=A} + \frac{1}{s}\int_{t=0}^{t=A} e^{-st}\,dt \right]
\end{aligned}$$

We can legally pull the term of $\frac{1}{s}$ outside of the integral in the last line because we are treating s as a constant throughout the integration. Evaluating the term $-\frac{t}{s}e^{-st}\big|_{t=0}^{t=A}$ leaves us with

$$\begin{aligned}
F(s) &= \lim_{A\to\infty} \left[-\frac{A}{s}e^{-sA} + \frac{1}{s}\int_{t=0}^{t=A} e^{-st}\,dt \right] \\
&= \lim_{A\to\infty} \left[-\frac{A}{s}e^{-sA} - \frac{1}{s^2}e^{-st}\Big|_{t=0}^{t=A} \right] \\
&= \lim_{A\to\infty} \left[-\frac{A}{s}e^{-sA} - \frac{1}{s^2}e^{-sA} + \frac{1}{s^2} \right]
\end{aligned}$$

Now since $s > 0$ we have that $\lim_{A\to\infty} e^{-sA} = 0$, so the limit of the second term is zero. Furthermore, we also have that $\lim_{A\to\infty} Ae^{-sA} = 0$ since, informally speaking, e^{-sA} goes to zero "faster" than A (or any polynomial expression in A) goes to infinity (one can make this argument formal by applying L'Hôpital's rule). Therefore we have that the limit of the first term is also zero. Therefore, for each $s > 0$, we obtain

$$F(s) = \lim_{A\to\infty} \left[-\frac{A}{s}e^{-sA} - \frac{1}{s^2}e^{-sA} + \frac{1}{s^2} \right] = 0 + 0 + \frac{1}{s^2}$$

Therefore the Laplace transform of $f(t) = t$ is defined for all $s > 0$ and is given for such s by

$$F(s) = \frac{1}{s^2}$$

Note that our final result is a function of the variable s. (In some sense we have just computed infinitely many integrals, one for each value of s). When computing Laplace transforms keep in mind that while doing the integration we treat s as though it were a constant, but that our final result will be a function of s. In particular, if you calculate a Laplace transform and get an answer that depends on t instead of (or in addition to) s, then you have made a mistake somewhere in your calculations.