

# An Informal Review of Matrices

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## 1 Introduction

Matrices play a major role in solving systems of ordinary differential equations, so it is necessary to familiarize oneself with some of the properties of matrices. This article is intended to supplement Section 7.2 of the textbook by presenting the material in a more informal manner.

## 2 Some Basic Definitions

A matrix is simply a rectangular array of (real or complex) numbers. We categorize matrices by how many rows they have, and how many columns they have. In particular, an  $m \times n$  matrix is a matrix with  $m$  rows and  $n$  columns.

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### 2.1 Example

If we define a matrix  $\mathbf{A}$  as below

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & 3 \\ 1 & 0 & -\frac{1}{2} \end{pmatrix}$$

Then  $\mathbf{A}$  is a  $2 \times 3$  matrix because it has 2 rows and 3 columns.

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There are two special kinds of matrices that we will find particularly useful. These are *square matrices* and *vectors*. A square matrix is an  $n \times n$  matrix, i.e. a matrix such that the number of rows is the same as the number

of columns. A column vector is an  $n \times 1$  matrix, i.e. a matrix with multiple rows but only one column. A row vector is a  $1 \times n$  matrix, i.e. a matrix with multiple columns but only one row.

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## 2.2 Example

Consider the following matrices  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$

$$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -3 \\ \pi \end{pmatrix}, \mathbf{C} = (3 \quad -4)$$

Then  $\mathbf{A}$  is a  $2 \times 2$  square matrix,  $\mathbf{B}$  is a  $3 \times 1$  column vector and  $\mathbf{C}$  is a  $1 \times 2$  row vector.

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Note that we often choose to denote vectors by lowercase letters such as  $\mathbf{x}$ .

## 3 Matrix Multiplication

Perhaps the simplest way to understand matrix multiplication is to first define an operation that we might call “multiplying a row by a column.”

### 3.1 Row-by-Column Multiplication

Consider the following row vector  $\mathbf{A}$  and column vector  $\mathbf{B}$  (note that both vectors have  $n$  entries)

$$\mathbf{A} = (a_1 \quad a_2 \quad \dots \quad a_n) \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix}$$

Then we define the product of  $\mathbf{A}$  and  $\mathbf{B}$  by  $\mathbf{AB} = a_1b_1 + a_2b_2 + \dots + a_nb_n$ . There are two things to note about this definition. The first is that the product of a row vector and a column vector is a number. The second is

that we only define the product of a row vector and a column vector if each vector has the same number of entries.

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### 3.1.1 Example

Let us define the following vectors

$$\mathbf{x} = (2 \quad -1 \quad 3), \mathbf{y} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}, \mathbf{z} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Then  $\mathbf{xy} = (2)(4) + (-1)(5) + (3)(1) = 6$ . On the other hand  $\mathbf{xz}$  is undefined because  $\mathbf{x}$  has three entries while  $\mathbf{z}$  only has two.

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We can apply this definition to the rows and columns of matrices as well

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### 3.1.2 Example

Consider the following two matrices.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 5 \\ 1 & -1 & -1 \end{pmatrix}$$

Suppose we wish to find the product of the second row of  $\mathbf{A}$  and the third column of  $\mathbf{B}$ . Then we would just note that the second row of  $\mathbf{A}$  is

$$(1 \quad -1 \quad 2)$$

while the third column of  $\mathbf{B}$  is

$$\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$$

Therefore the product of the second row of  $\mathbf{A}$  and the third column of  $\mathbf{B}$  is  $(1)(3) + (-1)(5) + (2)(-1) = -4$ . We can similarly find the product of any row of  $\mathbf{A}$  and any column of  $\mathbf{B}$ . On the other hand, we cannot define the product of any row of  $\mathbf{B}$  with any column of  $\mathbf{A}$  because the rows of  $\mathbf{B}$  have three entries while the columns of  $\mathbf{A}$  have two entries.

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## 3.2 The Product of Two Matrices

We are now ready to define the product of two matrices. Given two matrices  $\mathbf{A}$  and  $\mathbf{B}$ , we define the product  $\mathbf{AB}$  to be the matrix whose entry in the  $i$ th row and  $j$ th column is given by the product of the  $i$ th row of  $\mathbf{A}$  and the  $j$ th column of  $\mathbf{B}$ .

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### 3.2.1 Example

Consider the following two matrices.

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ 1 & -1 \end{pmatrix}$$

Suppose we wish to find the product  $\mathbf{AB}$ .

The entry in the first row and first column of  $\mathbf{AB}$  is the product of the first row of  $\mathbf{A}$  and the first column of  $\mathbf{B}$ , which is given by  $(-1)(3) + (0)(1) + (1)(1) = -2$ .

The entry in the first row and second column of  $\mathbf{AB}$  is the product of the first row of  $\mathbf{A}$  and the second column of  $\mathbf{B}$ , which is given by  $(-1)(2) + (0)(4) + (1)(-1) = -3$ .

The entry in the second row and first column of  $\mathbf{AB}$  is the product of the second row of  $\mathbf{A}$  and the first column of  $\mathbf{B}$ , which is given by  $(2)(3) + (-1)(1) + (2)(1) = 7$ .

The entry in the second row and second column of  $\mathbf{AB}$  is the product of the second row of  $\mathbf{A}$  and the second column of  $\mathbf{B}$ , which is given by  $(2)(2) + (-1)(4) + (2)(-1) = -2$ .

Putting this all together, we have that the product of  $\mathbf{A}$  and  $\mathbf{B}$  is

$$\mathbf{AB} = \begin{pmatrix} -2 & -3 \\ 7 & -2 \end{pmatrix}$$

On the other hand, similar computations would yield that the product of  $\mathbf{B}$  and  $\mathbf{A}$  is

$$\mathbf{BA} = \begin{pmatrix} 1 & -2 & 7 \\ 7 & -4 & 9 \\ -3 & 1 & -1 \end{pmatrix}$$

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## 4 Determinants

To any square matrix  $\mathbf{A}$  we can associate a number that we call the *determinant* of  $\mathbf{A}$ , denoted by  $\det \mathbf{A}$ . While we can define the determinant of any  $n \times n$  matrix for arbitrary  $n$ , for our purposes we only need to know the formulas for  $2 \times 2$  and  $3 \times 3$  matrices.

### 4.1 The Determinant of a $2 \times 2$ Matrix

Given a  $2 \times 2$  matrix  $\mathbf{A}$  as follows

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

we define the determinant of  $\mathbf{A}$  by

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

This calculation should look immediately familiar. It is exactly the same calculation we do whenever we need to find the Wronskian of two functions. Note that we often denote the determinant of a matrix by writing straight lines on the sides of the matrix instead of parentheses or brackets.

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#### 4.1.1 Example

Given the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -3 & 5 \end{pmatrix}$$

We can easily compute that the determinant of  $\mathbf{A}$  is

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 1 \\ -3 & 5 \end{vmatrix} = (2)(5) - (1)(-3) = 13$$

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## 4.2 The Determinant of a $3 \times 3$ Matrix

Given a  $3 \times 3$  matrix  $\mathbf{A}$  as follows

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

we define the determinant of  $\mathbf{A}$  by

$$\begin{aligned} \det(\mathbf{A}) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{aligned}$$

Though this formula may look very strange at first glance, there are a number of ways to remember it, two of which have been covered in lecture. Another way to remember the formula is to rewrite it in terms of  $2 \times 2$  determinants as follows

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

That is, we find the determinant of  $\mathbf{A}$  by taking the product of  $a_{11}$  with the determinant of a matrix formed by deleting the first row and first column of  $\mathbf{A}$ , subtracting the product of  $a_{12}$  with the determinant of a matrix formed by deleting the first row and second column of  $\mathbf{A}$  and then adding the product of  $a_{13}$  with the determinant of a matrix formed by deleting the first row and third column of  $\mathbf{A}$ . One can readily verify that this will indeed give the same result for the determinant of  $\mathbf{A}$  as the one we gave at the beginning of the section.

However we choose to remember how to compute determinants of  $3 \times 3$  matrices, we should note that the calculations are actually very similar to those we performed whenever we needed to calculate a cross product in Math 222 or Math 234.

### 4.2.1 Example

Given the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 4 \\ 1 & -2 & 2 \end{pmatrix}$$

We find that the determinant of  $\mathbf{A}$  is

$$\begin{aligned} \det(\mathbf{A}) &= \begin{vmatrix} 2 & 1 & 3 \\ -1 & 0 & 4 \\ 1 & -2 & 2 \end{vmatrix} \\ &= 2 \begin{vmatrix} 0 & 4 \\ -2 & 2 \end{vmatrix} - 1 \begin{vmatrix} -1 & 4 \\ 1 & 2 \end{vmatrix} + 3 \begin{vmatrix} -1 & 0 \\ 1 & -2 \end{vmatrix} \\ &= (2)[(0)(2) - (4)(-2)] - (1)[(-1)(2) - (4)(1)] + (3)[(-1)(-2) - (0)(1)] \\ &= 28 \end{aligned}$$

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