An Informal Review of Matrices, Part II

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1 Introduction

This handout is a continuation of "An Informal Review of Matrices" which covers inverses as well as matrix methods for solving systems of equations.

2 Inverse Matrices

In order to define the inverse of a matrix and come up with a practical method for computing it, we must first define the identity matrix and an augmented matrix.

2.1 The Identity Matrix

We denote by $\mathbf{I}_n$ the $n \times n$ *identity matrix*, i.e. the matrix with each *diagonal* entry equal to 1 and every other entry equal to 0. For example, the 2×2 identity matrix is given by

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

While the 3×3 identity matrix is given by

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Often, if it is clear from context what size matrices we are working with, we will omit the subscript n and simply write $\mathbf{I}$.

2.2 Augmented Matrices

Given two matrices $\mathbf{A}$ and $\mathbf{B}$, we define the *augmented matrix* $(\mathbf{A}|\mathbf{B})$ as the matrix where we place $\mathbf{A}$ on the left, $\mathbf{B}$ on the right, and place a vertical line to separate $\mathbf{A}$ and $\mathbf{B}$. We will refer to $\mathbf{A}$ as the left hand side of $(\mathbf{A}|\mathbf{B})$. Likewise, we will refer to $\mathbf{B}$ as the right hand side of $(\mathbf{A}|\mathbf{B})$.

2.2.1 Example

If we define the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as below

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Then the augmented matrices $(\mathbf{A}|\mathbf{B})$ and $(\mathbf{A}|\mathbf{C})$ are given by

$$(\mathbf{A}|\mathbf{B}) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 1 & 1 \\ 4 & 5 & 6 & 1 & 1 & 1 \\ 7 & 8 & 9 & 1 & 1 & 1 \end{array}\right)$$

$$(\mathbf{A}|\mathbf{C}) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{array}\right)$$

2.3 Finding Inverse Matrices

Given an $n \times n$ matrix $\mathbf{A}$, we define its inverse $\mathbf{A}^{-1}$ as the matrix which has the property that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$, if such a matrix exists. To compute $\mathbf{A}^{-1}$, we will use augmented matrices and the following three *row operations*

(1) Multiplying a row by a nonzero constant.

(2) Adding a constant multiple of one row to another row (we can also think of this as subtracting a constant multiple of one row from another).

(3) Swapping two rows.

These three operations help us to find inverses because of the following theorem.

Theorem. Suppose $\mathbf{A}$ is an $n \times n$ matrix and suppose that the augmented matrix $(\mathbf{A}|\mathbf{I}_n)$ can be transformed into the augmented matrix $(\mathbf{I}_n|\mathbf{B})$ by using the row operations (1), (2) and (3). Then $\mathbf{B} = \mathbf{A}^{-1}$.

We can use this theorem to develop a systematic procedure for finding the inverse of a given matrix.

2.3.1 A Step-by-Step Procedure for Computing the Inverse

Given an $n \times n$ matrix $\mathbf{A}$, we write out the augmented matrix $(\mathbf{A}|\mathbf{I}_n)$ and perform the following row operations to transform the left hand side into $\mathbf{I}_n$.

1. Make the entry in the first column and first row (i.e. the entry in the upper left hand corner) into a 1 by multiplying the first row by the appropriate constant. (You may need to swap the first row with a later row before you can do this).

2. Make the entry in the first column and second row into a 0 by adding or subtracting a constant multiple of the first row. Then make the entry in the first column and third row into a 0 by adding or subtracting a constant multiple of the first row. Continue in this fashion until we reach the last row of the matrix.

3. Make the entry in the second column and second row into a 1 by multiplying the second row by the appropriate constant. (You may need to swap the second row with a later row before you can do this).

4. Make the entry in the second column and first row into a 0 by adding or subtracting a constant multiple of the second row. Then make the entry in the second column and third row into a 0 by adding or subtracting a constant multiple of the second row. Continue in this fashion until we reach the last row of the matrix.

5. Continue this process (make the entry in the third column and third row into a one, then make all other entries in the thrid column into zeros ...) until one of the following two things happens:

(a) The left hand side becomes $\mathbf{I}_n$ so that we obtain an augmented matrix of the form $(\mathbf{I}_n|\mathbf{B})$. In this case we conclude that $\mathbf{B} = \mathbf{A}^{-1}$.

(b) We discover that we can't make the left hand side into $\mathbf{I}_n$, in which case we conclude that $\mathbf{A}$ is *singular* (i.e. $\mathbf{A}$ is not invertible).

If this procedure does not make sense to you, read through the following examples and then go over the procedure again.

2.3.2 Example

Find the inverse of the following matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 4 & 3 \\ 2 & 5 & 5 \end{pmatrix}$$

We now form the augmented matrix $(\mathbf{A}|\mathbf{I}_3)$ and attempt to transform it into the form $(\mathbf{I}_3|\mathbf{B})$. From now on we will write $\mathbf{I}$ instead of $\mathbf{I}_3$ since it should be clear from context that we are referring to the 3×3 identity matrix.

$$(\mathbf{A}|\mathbf{I}) = \left(\begin{array}{ccc|ccc} \mathbf{2} & 4 & 2 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 2 & 5 & 5 & 0 & 0 & 1 \end{array}\right)$$

We wish to make the entry in the first column and first row (indicated in bold) into a 1, and we do so by multiplying the first row by 1/2. Doing so transforms the first row from $(2\ 4\ 2|1\ 0\ 0)$ into $(1\ 2\ 1|\frac{1}{2}\ 0\ 0)$. So our augmented matrix becomes

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & \frac{1}{2} & 0 & 0 \\ \mathbf{1} & 4 & 3 & 0 & 1 & 0 \\ 2 & 5 & 5 & 0 & 0 & 1 \end{array}\right)$$

We now wish to make the entry in the first column and second row (again indicated in bold) into a 0. We do so by subtracting the first row from the second row. This will transform the second row from $(1\ 4\ 3|0\ 1\ 0)$ to $(0\ 2\ 2|-\frac{1}{2}\ 1\ 0)$. So our augmented matrix becomes

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 2 & 2 & -\frac{1}{2} & 1 & 0 \\ \mathbf{2} & 5 & 5 & 0 & 0 & 1 \end{array}\right)$$

We now wish to make the boldfaced entry of this matrix into a 0. We do so by subtracting two times the first row from the third row. This will transform the third row from $(2\ 5\ 5|0\ 0\ 1)$ into $(0\ 1\ 3|-1\ 0\ 1)$. So our augmented matrix becomes

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & \mathbf{2} & 2 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right)$$

We are now done with the first column and so we begin working on the second column. Our first task is to make the entry in the second column and second row (indicated in bold) into a 1. To do this we multiply the second row by 1/2, transforming our augmented matrix into

$$\left(\begin{array}{ccc|ccc} 1 & \mathbf{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right)$$

We now wish to make the boldfaced entry of this matrix into a 0. We do so by subtracting two times the second row from the first row, transforming our augmented matrix into

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \mathbf{1} & 3 & -1 & 0 & 1 \end{array}\right)$$

To complete work on the second column, we need to turn the boldfaced entry of this matrix into a 0. We do so by subtracting the second row from the third row, transforming our augmented matrix into

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & \mathbf{2} & -\frac{3}{4} & -\frac{1}{2} & 1 \end{array}\right)$$

We begin work on the third column by turning the boldfaced entry of this matrix into a 1. We do so by multiplying the third row by 1/2, transforming our augmented matrix into

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\mathbf{1} & 1 & -1 & 0 \\ 0 & 1 & 1 & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{3}{8} & -\frac{1}{4} & \frac{1}{2} \end{array}\right)$$

Our next task is to make the boldfaced entry of this matrix into a 0. We do so by adding the third row to the first row, transforming our augmented matrix into

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{8} & -\frac{5}{4} & \frac{1}{2} \\ 0 & 1 & \mathbf{1} & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{3}{8} & -\frac{1}{4} & \frac{1}{2} \end{array}\right)$$

Finally, we need to make the boldfaced entry of this matrix into a 0. We do so by subtracting the third row from the second row, transforming our augmented matrix into

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{8} & -\frac{5}{4} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{8} & \frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{8} & -\frac{1}{4} & \frac{1}{2} \end{array}\right)$$

The left hand side of this augmented matrix is $\mathbf{I}$, so we conclude that the right hand side is $\mathbf{A}^{-1}$. Thus we have found that

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{5}{8} & -\frac{5}{4} & \frac{1}{2} \\ \frac{1}{8} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{3}{8} & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

This next example illustrates a case where it is necessary to swap two rows before proceeding.

2.3.3 Example

Suppose we are asked to find the inverse of

$$\mathbf{A} = \begin{pmatrix} 0 & 3 & 1 \\ 3 & 2 & -1 \\ 4 & 3 & 5 \end{pmatrix}$$

Forming the augmented matrix $(\mathbf{A}|\mathbf{I})$ gives us

$$\left(\begin{array}{ccc|ccc} \mathbf{0} & 3 & 1 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 4 & 3 & 5 & 0 & 0 & 1 \end{array}\right)$$

We need to make the boldfaced entry into a 1. However, it is a 0, and there is nothing we can multiply 0 by to turn it into a 1. In this case we will instead start by swapping the first and second rows to obtain

$$\left(\begin{array}{ccc|ccc} 3 & 2 & -1 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 \\ 4 & 3 & 5 & 0 & 0 & 1 \end{array}\right)$$

(Don't forget to swap the right hand sides of the rows as well as the left hand sides!)

We then proceed as before by making the entry in the first column and first row into a 1 by multiplying the first row by 1/3 and then continuing from there.

In general, if you need to create a 1 where you currently have a 0, swap the row you're working on with one of the rows below it.

This next example illustrates a case where the matrix is singular.

2.3.4 Example

Find the inverse of the following matrix, if it exists

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & -1 & 5 \end{pmatrix}$$

Forming the augmented matrix $(\mathbf{A}|\mathbf{I})$ gives us

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ 0 & -1 & 5 & 0 & 0 & 1 \end{array}\right)$$

Subtracting two times the first row from the second row gives us

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & -1 & 5 & 0 & 0 & 1 \end{array}\right)$$

Note that the left hand side of the second row consists entirely of zeros. At this point we can stop and conclude that the matrix is singular. Any attempt to transform the left hand side into $\mathbf{I}$ will fail. Similarily, if we obtain a column of all zeros on the left hand side, we can immediately conclude that the matrix is singular.

It is also possible to show that a matrix is singular before attempting to solve it. This is because a square matrix is singular if and only if its determinant is zero. So to determine whether a square matrix is singular or invertible beforehand, we can simply compute the determinant.

3 Matrix Methods for Solving Systems of Linear Equations

Until now, when given a system of linear equations, we have found solutions by directly manipulating the given equations. We can also use matrices to solve such equations in a systematic manner.

3.1 Representing a System of Equations a Matrix

Suppose we are given a system of equations as follows.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

Then we can represent this system as a matrix equation by writing

$$\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

One can see that this represents the above system of equations by multiplying out the two matrices on the left. We can further represent this system as the following augmented matrix

$$\left(\begin{array}{ccc|c} a_{11} & \ldots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \ldots & a_{nn} & b_n \end{array} \right)$$

Once we represent a system of equations as an augmented matrix, we can solve the system by performing row operations on the augmented matrix, then converting back to a system of equations. The process is largely similar to what we did to find the inverse of a given matrix, although somewhat easier because we can skip a few steps that were necessary when finding inverses.

3.2 Solving a System of Linear Equations Using Matrices

We can outline a step-by-step procedure for solving a system of equations using matrices, but since it is so similar to the process of inverting a matrix we will instead illustrate the process by example, pointing out where it differs from finding inverses along the way.

3.2.1 Example

Solve the following system of equations.

$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &= 8 \\ x_1 + 4x_2 + x_3 &= 6 \\ -x_1 + x_2 + 2x_3 &= 6 \end{aligned}$$

We first proceed by setting up the augmented matrix

$$\left(\begin{array}{ccc|c} \mathbf{2} & 4 & -6 & 8 \\ 1 & 4 & 1 & 6 \\ -1 & 1 & 2 & 6 \end{array}\right)$$

We turn the boldfaced term into a 1 by multiplying the first row by 1/2. Then our augmented matrix becomes

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ \mathbf{1} & 4 & 1 & 6 \\ \mathbf{-1} & 1 & 2 & 6 \end{array}\right)$$

We now need to turn each of the boldfaced terms into zeros. We can do this by first subtracting the first row from the second row, and then adding the first row to the third row. After performing these two row operations our augmented matrix will become

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & \mathbf{2} & 4 & 2 \\ 0 & 3 & -1 & 10 \end{array}\right)$$

Multiplying the second row by 1/2 to turn the boldfaced term into a 1 will give us an augmented matrix of

$$\left(\begin{array}{ccc|c} 1 & \mathit{2} & -3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & \mathbf{3} & -1 & 10 \end{array}\right)$$

At this point if we were trying to find an inverse we would try to turn both the italicized term and the boldfaced term of this augmented matrix into zeros. It will turn out, however, that when solving a system of equations we can leave the italicized term alone and still get the solution. So, in the interest of saving ourselves some time, we will only worry about making the boldfaced term into a 0. Subtracting three times the second row from the third row, our augmented matrix becomes

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -\mathbf{7} & 7 \end{array}\right)$$

We then make the boldfaced term into a 1 by multiplying the third row by -1/7, giving us an augmented matrix of

$$\left(\begin{array}{ccc|c} 1 & 2 & -\mathit{3} & 4 \\ 0 & 1 & \mathit{2} & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Here the italics are again used to indicate terms that we would turn into zeros if we were inverting a matrix, but that we will leave alone in this case. In general, when solving systems of equations in this way, once we create a 1 in the appropriate location in a column we want to turn every term that appears below it in that column into a 0, while we are free to leave the terms above it alone.

In the augmented matrix above we now have a 1 in each diagonal entry of the left hand side. At this point we will now turn our augmented matrix back into a system of equations and now solve that system (which is equivalent to what we started with but easier to solve). In our case we obtain

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= \ \ 4 \\ x_2 + 2x_3 &= \ \ 1 \\ x_3 &= -1 \end{aligned}$$

The third equation immediately gives us $x_3 = -1$. Plugging -1 in for x_3 in the second equation gives us $x_2 = 3$. Then plugging 3 in for x_2 and -1 in for x_3 into the first equation gives us $x_1 = -5$. Therefore we have solved the system.

The idea here is to get the augmented matrix into *upper triangular form*, where each diagonal entry is 1 and all the entries below are 0. Once we have done so, we will end up with an easy system of equations as in the above example. However, sometimes we can stop our process even before we reach that point. The next example illustrates such a case.

3.2.2 Example

Solve the following system of equations

$$\begin{aligned} x_1 - \ \ x_2 + 2x_3 &= 0 \\ x_1 + \ \ x_2 + 2x_3 &= 2 \\ -x_1 + 2x_2 - 4x_3 &= 1 \end{aligned}$$

We represent this system by the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 1 & 1 & 2 & 2 \\ -1 & 2 & -4 & 1 \end{array}\right)$$

The upper left hand term is already a 1, so we move on to the next step of making each terms below it into a 0. Subtracting the first row from the second row and then adding the first row to the third rwo gives us

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

Rather than continue to work on this matrix, we should see that we are already at a point where we can find the solution. Translating back to equations, we have

$$\begin{aligned} x_1 - \ x_2 + 2x_3 &= 0 \\ 2x_2 \qquad &= 2 \\ x_2 - 2x_3 &= 1 \end{aligned}$$

The second equation gives us $x_2 = 1$. Plugging in 1 for x_2 in the third equation will give us that $x_3 = 0$. Plugging in 1 for x_2 and 0 for x_3 in the first equation will give us $x_1 = 1$.

A good way to think about this is that after you have performed a row operation, you should glance at the augmented matrix you have left and see whether the solution to the system is clear. Once we do reach the point where the solution is clear, we can stop working with the augmented matrix.

Sometimes we can identify when a system has no solution at some point while working with the augmented matrix.

3.2.3 Example

Solve the following system of equations.

$$\begin{aligned} x_1 - \ x_2 + 2x_3 &= 0 \\ x_1 - \ x_2 \qquad &= 2 \\ -2x_1 + 2x_2 - 4x_3 &= 1 \end{aligned}$$

We represent this system as the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ -2 & 2 & -4 & 1 \end{array}\right)$$

The upper left hand term is already a 1, so we continue on to the step of making each term below it into a 0. Subtracting the first row from the second row and adding two times the first row to the third row yields

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

This represents the system of equations

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ -2x_3 &= 2 \\ 0 &= 1 \end{aligned}$$

The third equation is clearly a false statement. We have reached a contradiction, and therefore no solution exists.

In general, if we obtain a row of zeros on the left hand side, and the right hand side of that row is some *nonzero* number a then no solution exists because that row represents the false statement $0 = a$.

WARNING! If at some point while solving the system you obtain a row with all zeros on the left hand side *and* the right hand side of that row is also 0, do **NOT** conclude that no solution exists. In that case the row simply represents the *true* statement $0 = 0$, so we have not reached a contradiction. A solution may still exist and we should continue looking for it.

The following example illustrates the situation where a solution exists, but is not unique.

3.2.4 Example

Solve the following system of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + 2x_2 - x_3 &= 4 \\ 2x_1 + 3x_2 \phantom{{}-x_3} &= 1 \end{aligned}$$

We represent this system as the following augmented matrix.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \\ 2 & 3 & 0 & 7 \end{array}\right)$$

The upper left hand term is already 1, so we move on to the step of making each term below it into 0. Subtracting the first row from the second row and then subtracting two times the first row from the third row transforms our augmented matrix into

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

This represents the system of equations

$$\begin{aligned} x_1 + x_2 + \quad x_3 &= 3 \\ x_2 - 2x_3 &= 1 \\ x_2 - 2x_3 &= 1 \end{aligned}$$

The second and third equations are redundant, so we can simplify this to

$$\begin{aligned} x_1 + x_2 + \quad x_3 &= 3 \\ x_2 - 2x_3 &= 1 \end{aligned}$$

Here we have two equations but three unknowns, so there is no way we will have a unique solution. In fact, there are infinitely many solutions. To see this we can let $x_3 = c$, where c is an arbitrary constant. Then plugging in c for x_3 in the second equation gives us $x_2 = 1 + 2c$. Plugging in $1 + 2c$ for x_2 and c for x_3 gives us $x_1 = 2 - 3c$. Therefore we have as a solution

$$\begin{aligned} x_1 &= 2 - 3c \\ x_2 &= 1 + 2c \\ x_3 &= c \end{aligned}$$

This is a solution for *any* choice of the constant c. We can then obtain specific solutions by choosing some value for c. For instance, letting $c = 0$ gives the solution

$$
\begin{aligned}
x_1 &= 2 \\
x_2 &= 1 \\
x_3 &= 0
\end{aligned}
$$

On the other hand, letting $c = 1$ gives the solution

$$
\begin{aligned}
x_1 &= -1 \\
x_2 &= 3 \\
x_3 &= 1
\end{aligned}
$$

When solving a system of equations, one thing to keep in mind is that sometimes it's faster to work directly with the equations, while other times it's faster to use matrices. In general, if we have only two equations, then it is faster to simply work with the equations themselves. If there are three or more equations, then it is often the case that using matrices will save us some time.