# A Brief Overview of Partial Derivatives

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## 1 Introduction

It is occasionally necessary to calculate a partial derivative in Math 319. The purpose of this article is for you to familiarize yourself with how to calculate them. This is intended to be a brief overview, not an extensive treatment, so I will not be bothering with formal definitions, applications or the like. A much more extensive treatment of partial derivatives and their applications is given in Math 234.

## 2 Informal Definition and Notation

Given a (sufficiently nice) function of two variables $f(x, y)$, we can define a function known as the **partial derivative of $f$ with respect to** $x$, which we will denote by either $f_x$ or $\frac{\partial f}{\partial x}$. We calculuate $f_x$ by regarding $f$ as a function of $x$ alone and treating $y$ as a constant, then taking the derivative of that function. We can similarily define the **partial derivative of $f$ with respect to** $y$ (similarily denoted $f_y$ or $\frac{\partial f}{\partial y}$) by differentiating $f$ as a function of $y$ alone, with $x$ held constant. One can also define partial derivatives of functions of three (or more) variables. For instance, if $g(x, y, z)$ is a function of three variables, then we calculate $g_x$ by holding $y$ and $z$ constant and regarding $g$ as a function of $x$ alone.

## 3 Some Examples

**Example 1.** Let $f(x, y) = \sin x + \cos y$. Then $f_x = \cos x$ and $f_y = -\sin y$.

Note that the $\cos y$ term dropped out of $f_x$ since it did not depend on $x$, and similarily the $\sin x$ term dropped out of $f_y$ because it did not depend on $y$.

It may be helpful to think of finding $f_x$ as being like finding the derivative of $f$ after replacing all the $y$'s with some constant, say 2. If we were calculating the derivative of $\sin x + \cos 2$, we would get that the $\cos 2$ term drops out because it is a constant, and our answer would be $\cos x$. Similarly, mentally replacing all the $x$'s with some constant, say $e$, and then differentiating can be helpful in thinking about what $f_y$ is.

**Example 2.** Let $g(x, y) = x^2 e^{-y}$. Then $\frac{\partial g}{\partial x} = 2xe^{-y}$ and $\frac{\partial g}{\partial y} = -x^2 e^{-y}$.

The result for $\frac{\partial g}{\partial x}$ follows from the fact that the derivative of $x^2$ is $2x$, while the $e^{-y}$ term is regarded as constant. The result for $\frac{\partial g}{\partial y}$ follows from the fact that the derivative of $e^{-y}$ is $-e^{-y}$ while the $x^2$ term is regarded as constant.

Again, replacing $y$ by a constant such as $\pi$ can be helpful when dealing with $\frac{\partial f}{\partial x}$. That is, the problem of finding $\frac{\partial f}{\partial x}$ is similar to the problem of finding the ordinary derivative of the function $x^2 e^{-\pi}$. Likewise finding $\frac{\partial f}{\partial y}$ is similar to finding the ordinary derivative of $\pi^2 e^{-y}$.

**Example 3** Let $u(x, y) = \sin(xy)$. Then $u_x = y\cos(xy)$ and $u_y = x\cos(xy)$.

Here the chain rule comes into play. To calculate $u_x$ we first differentiate sine and then multiply by $y$ because $y$ is the partial derivative of $xy$ with respect to $x$.

This is similar to calculating the ordinary derivative of the single variable function $\sin(2x)$. There we would differentiate sine and then multiply by 2 because 2 is the derivative of $2x$ to get that the derivative of $\sin(2x)$ is $2\cos(2x)$.

**Example 4.** Let $M(x, y) = x^2 y + x - y + 2$. Then $\frac{\partial M}{\partial x} = 2xy + 1$ and $\frac{\partial M}{\partial y} = x^2 - 1$.

The result for $\frac{\partial M}{\partial x}$ follows from the fact that the partial derivative of $x^2 y + x$ with respect to $x$ is $2xy + 1$ (remember, the $y$ is treated as a constant), while the $-y+2$ drops out because it doesn't depend on $x$. Similarily, the result for $\frac{\partial M}{\partial y}$ follows from the fact that the partial derivative of $x^2 y - y$ with respect to $y$ is $x^2 - 1$ (again, we treat $x$ as a constant) and the $x + 2$ drops out because it doesn't depend on $y$.

# 4 Practice Problems

The following problems are for practice at calculating partial derivatives. Solutions are at the bottom of the page.

**1.** Find $f_x$ and $f_y$ for $f(x, y) = x \sin y$

**2.** Find $u_x$ and $u_y$ for $u(x, y) = x\sqrt{1 + y^2} + 2x$

**3.** Find $\frac{\partial M}{\partial x}$ and $\frac{\partial M}{\partial y}$ for $M(x, y) = xe^{xy^2}$

**Solutions**

**1.** $f_x = \sin y$ and $f_y = x \cos y$

**2.** $u_x = \sqrt{1 + y^2} + 2$ and $u_y = \frac{xy}{\sqrt{1+y^2}}$

**3.** $\frac{\partial M}{\partial x} = e^{xy^2} + xy^2 e^{xy^2}$ and $\frac{\partial M}{\partial y} = 2x^2 y e^{xy^2}$