Math 217 - Spring 2008 - Quiz 4

NAME: SOLUTIONS

1. (5 points) Can there be a continuous function on $[0,1]$ such that the n-th upper sum, U_n, and the n-th lower sum, L_n, are given by

$$L_n = \frac{n^2 - n + 3}{n^2 + 7n - 1} \quad \text{and} \quad U_n = \frac{2 + \cos(\sqrt{2}n)}{4n^3 - n^2 + 7},$$

for any n?

Solution: No. Indeed, if there was such a function f then the upper an lower sums would both converge to area under the curve. That is,

$$\lim_{n\to\infty} U_n = \int_0^1 f(x)\,dx = \lim_{n\to\infty} L_n.$$

However, $U_n \to 0$ as $n \to \infty$ and $L_n \to 1$ as $n \to \infty$. The above equality implies $0 = 1$, a contradiction, so no such function exists.

2. (5 points) Find the value of

$$\int_2^5 (x^3 + 3x^2)\,dx$$

by using Riemann sums.

Solution: Consider the partition of $[2,5]$:

$$\mathcal{P} = \left\{2, 2 + \frac{3}{n}, 2 + \frac{6}{n}, \ldots, 2 + \frac{3n}{n} = 5\right\}.$$

The right endpoint sums corresponds to the upper sum over this partition; hence,

$$U(f,\mathcal{P}) = \sum_{j=1}^{n} \frac{3}{n} f\Big(2 + \frac{3j}{n}\Big) = \frac{3}{n}\sum_{j=1}^{n}\left[\Big(2+\frac{3j}{n}\Big)^3 + 3\Big(2 + \frac{3j}{n}\Big)^2\right] =$$

$$\frac{3}{n}\sum_{j=1}^{n}\left[\left(8 + 12\Big(\frac{3j}{n}\Big) + 6\Big(\frac{3j}{n}\Big)^2 + \Big(\frac{3j}{n}\Big)^3\right) + \left(12 + 12\Big(\frac{3j}{n}\Big) + 3\Big(\frac{3j}{n}\Big)^2\right)\right] =$$

$$\frac{3}{n}\sum_{j=1}^{n}\left[20 + \frac{72j}{n} + \frac{81j^2}{n^2} + \frac{27j^3}{n^3}\right] = \frac{60}{n}\sum_{j=1}^{n} 1 + \frac{216}{n^2}\sum_{j=1}^{n} j + \frac{243}{n^3}\sum_{j=1}^{n} j^2 + \frac{81}{n^4}\sum_{j=1}^{n} j^3 =$$

$$60 + \frac{216}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{243}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2.$$

Now as $\|\mathcal{P}\| \to 0$ we necessarily have $n \to \infty$. So, passing to the limit we have

$$\lim_{\|\mathcal{P}\|\to 0} U(f,\mathcal{P}) = \lim_{n\to\infty}\left(60 + \frac{216}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{243}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2\right) =$$

$$60 + 108 + 81 + \frac{81}{4} = \frac{1077}{4}.$$

(Hopefully you have fallen more in love with the Fundamental Theorem of Calculus after that computation.)

3. (5 points) Find the value of the following limit:

$$\lim_{n\to\infty} \frac{1}{n}\Big(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\Big).$$

Solution: In Σ notation the question is equivalent to finding the following limit:

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n}\sqrt{\frac{j}{n}}.$$

However, it is easy to see that this is the upper Riemann sum of the function $f(x) = \sqrt{x}$ with respect to the partition $\mathcal{P} = \{0, 1/n, 2/n, 3/n, \ldots, n/n = 1\}$. Since this function is continuous on the set $[0, 1]$, we conclude that the limit of the upper sums as $n \to \infty$ is convergent to the definite integral

$$\int_0^1 \sqrt{x}\, dx = \frac{2}{3}x^{3/2}\Big]_0^1 = \frac{2}{3}.$$

4. (5 points) Find the following limit

$$\lim_{x\to 3} \Big(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}\, dt\Big).$$

Solution: Plugging 3 into the expression, we see that the limit has a "0/0" form. Applying L'Hôpital's, we see

$$\lim_{x\to 3} \Big(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}\, dt\Big) = \lim_{x\to 3} \frac{x\Big(\frac{\sin x}{x}\Big) + \int_3^x \frac{\sin t}{t}\, dt}{1} = \sin 3,$$

where we used the Fundamental Theorem of Calculus and the product rule when taking the derivative of the numerator.