Math 217 - Spring 2008 - Quiz 5

NAME: SOLUTIONS

Instructions: Answer each of the following questions using complete sentences either in terms of mathematical expressions or the English language equivalent.

1. (5 points) Suppose that the graph of f passes through the origin and the point $(1, 1)$. Find

$$\int_0^1 f'(x)\,dx.$$

Solution: The Fundamental Theorem of Calculus implies

$$\int_0^1 f'(x)\,dx = f(1) - f(0) = 1 - 0 = 1.$$

2. (5 points) Given the function

$$f(x) = \tan x \int_x^{\sin x} \frac{1}{\sqrt{1+t^5}}\,dt.$$

Find $f'(x)$.

Solution: Using the product and Leibnitz rules we have

$$f'(x) = \sec^2 x \int_x^{\sin x} \frac{1}{\sqrt{1+t^5}}\,dt + \tan x \Big[\frac{\cos x}{\sqrt{1+\sin^5 x}} - \frac{1}{\sqrt{1+x^5}}\Big].$$

3. (5 points) Find values of c such that the area enclosed by the regions $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576.

Solution: Finding the intersection points of the two curves establishes the limits of integration: $c^2 - x^2 = x^2 - c^2$ implies that $x^2 = c^2$ and hence $x = \pm c$. The area in between the curves is given by

$$A = \int_{-c}^{c} [(c^2 - x^2) - (x^2 - c^2)]\,dx = \int_{-c}^{c} (2c^2 - 2x^2)\,dx = 2c^2 x - \frac{2}{3}x^3\Big|_{-c}^{c} = \frac{8}{3}c^3.$$

By assumption $576 = A$, so solving the equation $576 = 8c^3/3$, we conclude that $c = 6$.

4. (5 points) Prove that

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx,$$

for any f continuous by making a change of variables.

Solution: Make the change of variables $u = \pi - x$, from which we conclude that $du = -dx$ and the first integral becomes

$$\int_0^\pi xf(\sin x)\,dx = \int_\pi^0 (\pi - u)f(\sin(\pi - u))\,(-du) = \int_0^\pi (\pi - u)f(\sin u)\,du =$$

$$\pi \int_0^\pi f(\sin u)\,du - \int_0^\pi uf(\sin u)\,du.$$

Adding the last expressions to both sides we get

$$2\int_0^\pi xf(\sin x)\,dx = \pi \int_0^\pi f(\sin u)\,du \quad \Rightarrow \quad \int_0^\pi xf(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$$