

# LENGTHS OF FINITE DIMENSIONAL REPRESENTATIONS OF PBW ALGEBRAS

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ABSTRACT. Let  $\Sigma$  be a set of  $n \times n$  matrices with entries from a field, for  $n > 1$ , and let  $c(\Sigma)$  be the maximum length of products in  $\Sigma$  necessary to linearly span the algebra it generates. Bounds for  $c(\Sigma)$  have been given by Paz and Pappacena, and Paz conjectures a bound of  $2n - 2$  for any set of matrices. In this paper we present a proof of Paz's conjecture for sets of matrices obeying a modified Poincaré-Birkhoff-Witt (PBW) property, applicable to finite dimensional representations of Lie algebras and quantum groups. A representation of the quantum plane establishes the sharpness of this bound, and we prove a bound of  $2n - 3$  for sets of matrices which do not generate the full algebra of all  $n \times n$  matrices. This bound of  $2n - 3$  also holds for representations of Lie algebras, although we do not know whether it is sharp in this case.

## 1. INTRODUCTION

For a fixed integer  $n > 1$ , let  $\Sigma = \{X_1, \dots, X_t\}$  be a set of  $n \times n$  matrices over an arbitrary field  $\mathbf{k}$ , and let  $\Sigma^m$  be the set of products of length  $m$  in the  $X_i$ , where  $\Sigma^0$  is defined as the identity. Let  $L_i$  be the linear space spanned by  $\Sigma^0 \cup \Sigma^1 \cup \dots \cup \Sigma^i$ , and denote the dimension of this space by  $r_i$ . Next, let  $L_*$  be the linear space spanned by products of any length, and let  $r_*$  denote its dimension. Finally, let  $c(\Sigma) = \min\{i : r_i = r_*\}$ .

In [3], Paz proved that  $c(\Sigma) \leq \lceil (n^2 + 2)/3 \rceil$ , and Pappacena gave lower bounds [2]. Paz conjectured a bound of  $2n - 2$  and suggested a lemma which, if proved, would prove the conjecture. We prove this lemma (listed as Proposition 2.5 below) for matrices satisfying the following property: every product  $u = X_{i_1} X_{i_2} \cdots X_{i_l}$  in the matrices  $X_1, \dots, X_t$  can be written, modulo  $L_{l-1}$ , in the form

$$\sum_{j_1 + j_2 + \cdots + j_t = l} c_{(j_1, \dots, j_t)} X_t^{j_t} X_{t-1}^{j_{t-1}} \cdots X_1^{j_1},$$

with  $c_{(j_1, \dots, j_t)} = 0$  whenever  $X_t^{j_t} X_{t-1}^{j_{t-1}} \cdots X_1^{j_1} < u$  in the lexicographical ordering.

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This modified PBW property allows any  $l$ -length matrix product  $u$  to be written as a linear combination of ordered products of length  $l$ , modulo products (not necessarily ordered) of lesser length; here by *ordered product* we mean a product in which  $X_i X_j$  never appears for  $i < j$ . Our condition is, in fact, looser than what is generally found in homomorphic images of algebras satisfying the PBW property. Sets of matrices obeying this property include finite dimensional representations of Lie algebras and quantum groups (see, e.g., [1] for further details).

The  $2n - 2$  bound is in fact sharp for the class of matrices satisfying the above property, as an example using the quantum plane illustrates, but lower bounds for certain other cases can be obtained. In particular, knowledge about  $r_*$  allows the constraints of Paz's suggested lemma to be tightened, resulting in lower bounds on  $c(\Sigma)$ . As an example we provide a proof of a  $2n - 3$  bound when the  $\mathbf{k}$ -algebra generated by  $\Sigma$  is not equal to  $M_n(\mathbf{k})$ , the full algebra of all  $n \times n$  matrices over the field  $\mathbf{k}$ .

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## 2. PROOF OF THE MAIN THEOREM

We begin with some notation and preliminaries necessary to our proof. We then prove Lemmas 2.3 and 2.4. Together, these preliminary lemmas establish Paz's suggested lemma, listed below as Proposition 2.5. We then proceed to prove our main theorem.

### 2.1 Notation.

(i) We write the matrix product  $X_{i_1} \cdots X_{i_k}$  as the word  $i_1 \dots i_k$ . An  $l$ -subword of a word  $u$  is any set of  $l$  consecutive letters in  $u$ .

(ii) We say two words are *formally equivalent* if their  $i^{\text{th}}$  letters match for all  $i$ ; otherwise we say they are *formally distinct*. We call a subword consisting of one repeated letter, such as  $111 \dots 1$ , *formally constant*.

(iii) If  $u_i, u_j, u_k, \dots, u_p$  are  $m$ -length products,  $u_i \times (u_j, u_k, \dots, u_p)$  means  $u_i$  is a linear combination of  $u_j, u_k, \dots, u_p$  modulo  $L_{m-1}$ .

(iv) We call a word *reducible* if it can be written as a linear combination of words of lesser length.

### 2.2 Preliminaries.

(i) Any word that can be written as a linear combination, modulo words of lesser length, of other words that are all reducible is itself reducible. Thus, we will examine only ordered products, and the modified PBW property satisfied by the matrices we are considering guarantees that our results will carry over to all matrix products.

(ii) If any word contains a subword of  $n$  or more of the same letter this word will be reducible by the Cayley-Hamilton Theorem.

(iii) Any word  $u$  can be treated as a base- $(t+1)$  number, and this number will be unique to  $u$ . We denote this number by  $\bar{u}$ . The numerical ordering on  $\bar{u}$  coincides exactly with the lexicographical ordering on matrix products.

**Lemma 2.3.** *For any positive integers  $k, m$  and  $N$ , if  $r_k - r_{k-1} \leq N$  then any word of length  $m$  with more than  $N$  formally distinct  $k$ -subwords can be written as a linear combination, modulo  $L_{m-1}$ , of words each having at most  $N$  formally distinct  $k$ -subwords.*

*Proof.* Given such a word  $u$  of length  $m$ , let  $u_1, u_2, \dots, u_s$  for  $s > N$  be  $u$ 's formally distinct  $k$ -subwords, numbered such that  $\bar{u}_1 < \bar{u}_2 < \dots < \bar{u}_s$ . Since there are more subwords than  $r_k - r_{k-1}$ , these subwords must be linearly dependent, modulo  $L_{k-1}$ . Therefore there exists a minimum  $i$  such that  $u_i \propto (u_{k_1}, u_{k_2}, \dots, u_{k_p})$  with  $k_1, k_2, \dots, k_p > i$ . (Note that if  $i = s$  then  $u_s$  is equal, modulo  $L_{k-1}$ , to zero, and  $u$  is trivially reducible.) We form the new words  $u^{(1)}, u^{(2)}, \dots, u^{(p)}$  from  $u$  by replacing  $u_i$  with  $u_{k_l}$  to form  $u^{(l)}$ . We see that  $u \propto (u^{(1)}, u^{(2)}, \dots, u^{(p)})$  and that  $\bar{u}^{(l)} > \bar{u}$  for all  $l$ .

We can apply the above process to each of the words  $u^{(l)}$  as long as they have more than  $N$  formally distinct subwords. Since this process continually increases the numerical value of these words and there are finitely many words of length  $m$ , we will eventually write  $u$  as a linear combination, modulo  $L_{m-1}$ , of words with at most  $N$  formally distinct  $k$ -subwords.

In addition, because rewriting words in ordered form via the modified PBW property also continually increases their numeric value, we can, at each step in the above process, put all our words in ordered form. Therefore, when working with sets of matrices obeying the modified PBW property we can write any ordered word  $u$  as a linear combination, modulo  $L_{m-1}$ , of *ordered* words with at most  $N$  formally distinct  $k$ -subwords.  $\square$

**Lemma 2.4.** *For a positive integer  $k \leq 2n - 2$ , set*

$$N = \begin{cases} k & \text{for } 1 \leq k \leq n - 1 \\ 2n - k - 2 & \text{for } n \leq k \leq 2n - 2. \end{cases}$$

*Any ordered word of length  $2n - 1$  not reducible by the Cayley-Hamilton Theorem contains at least  $N + 2$  formally distinct  $k$ -subwords.*

*Proof.* Let  $u$  be an ordered word of length  $2n - 1$  that is not reducible by the Cayley-Hamilton Theorem.

*Case I.* Suppose  $1 \leq k \leq n - 1$ . Recall  $N = k$ .

*Subcase i.* The longest formally constant subword in  $u$  has length greater than or equal to  $k$ .

Specifically, call the longest formally constant subword  $w$  and say it has length  $j \geq k$ . Since our word is not reducible by Cayley-Hamilton,  $j < n$ . Since  $k \leq j < n$  and  $u$  has length  $2n - 1$  there will be at least  $k + 1 = N + 1$   $k$ -subwords overlapping but not contained in  $w$ . Examining these  $k$ -subwords, we see that they will be formally distinct since each features the transition between  $w$  and the surrounding letters in a different spot. Thus, including one of the formally constant  $k$ -subwords found within  $w$ , we conclude that  $u$  contains at least  $N + 2$  formally distinct  $k$ -subwords.

*Subcase ii.* The longest formally constant subword in  $u$  has length less than  $k$ .

In this case no two  $k$ -subwords will be formally equivalent since none will be constant. Since  $k < n$ ,  $u$  has at least  $n + 1$   $k$ -subwords. Since  $N = k < n$ ,  $u$  contains at least  $N + 2$  formally distinct  $k$ -subwords.

*Case II.* Now suppose  $n \leq k \leq 2n - 2$ . Recall  $N = 2n - 2 - k$ . Since  $k \geq n$ , no two  $k$ -subwords can be formally equivalent or else  $u$  will be reducible by Cayley-Hamilton since  $u$  will contain a formally constant subword of length greater than  $n$ . There are  $2n - k$   $k$ -subwords in total, so  $u$  contains  $N + 2$  formally distinct  $k$ -subwords.  $\square$

We now prove the lemma suggested by Paz.

**Proposition 2.5.** *Let  $m = 2n - 1$ . If for some positive integer  $k \leq 2n - 2$  the corresponding condition from among*

$$r_k - r_{k-1} \leq k \text{ for } 1 \leq k \leq n - 1$$

$$r_k - r_{k-1} \leq 2n - k - 2 \text{ for } n \leq k \leq 2n - 2$$

*holds then  $c(\Sigma) \leq m - 1$ .*

*Proof.* Lemmas 2.3 and 2.4 establish this proposition in the following manner. Suppose one of the above conditions holds; say it is the condition for  $k^*$  and let  $N^*$  correspond to  $k^*$  as described in the statement of Lemma 2.4. Consider a word  $u$  of length  $m = 2n - 1$ . We will show that  $u$  is reducible, giving us that  $c(\Sigma) \leq m - 1 = 2n - 2$ .

If  $u$  is reducible by Cayley-Hamilton, we are done. If  $u$  is not reducible by Cayley-Hamilton then Lemma 2.4 implies that it has more than  $N^*$  formally distinct  $k^*$ -subwords. Lemma 2.3 then implies that  $u$  is linearly dependent, modulo  $L_{m-1}$ , on ordered words which do not have more than  $N^*$  distinct  $k^*$ -subwords. Finally, the contrapositive of Lemma 2.4 implies that these words are reducible by Cayley-Hamilton. Thus  $u$  is reducible, and  $c(\Sigma) \leq m - 1$ .  $\square$

We now prove our main theorem.

**Theorem 2.6.** *Let  $\Sigma = \{X_1, \dots, X_t\}$  be a set of  $n \times n$  matrices satisfying the following property: any product  $X_{i_1} \cdots X_{i_l}$  can be written, modulo  $L_{l-1}$ , in the form*

$$\sum_{j_1 + \cdots + j_t = l} c_{(j_1, \dots, j_t)} X_t^{j_t} \cdots X_1^{j_1}, \text{ with } c_{(j_1, \dots, j_t)} = 0 \text{ whenever } \overline{X_t^{j_t} \cdots X_1^{j_1}} < \bar{u}.$$

*Then  $c(\Sigma) \leq 2n - 2$ .*

*Proof (following Paz [3]).* If  $c(\Sigma) = m \geq 2n - 1$ , none of the conditions of Proposition 2.5 can hold. Thus, if  $c(\Sigma) \geq 2n - 1$ , then  $r_0 = 1, r_1 - r_0 \geq 2, r_2 - r_1 \geq 3, \dots, r_{n-1} - r_{n-2} \geq n, r_n - r_{n-1} \geq n - 1, \dots, r_{2n-2} - r_{2n-3} \geq 1$ . Then we have  $r_{2n-2} \geq 1 + 2 + \cdots + n - 1 + n + n - 1 + \cdots + 1 = 2(n(n - 1))/2 + n = n^2 \geq r_*$ . This, however, contradicts  $c(\Sigma) \geq 2n - 1$ , for if  $r_{2n-2}$  is already greater than or equal to the dimension of all of  $L_*$ ,  $r_{2n-1}$  can be no larger than  $r_{2n-2}$ .  $\square$

### 3. SHARPNESS OF THE BOUND

The bound of  $2n - 2$  is sharp for the general set of matrices described above as the following example from the quantum plane shows.

Consider complex  $n \times n$  matrices  $X$  and  $Y$  satisfying  $XY = qYX$ , where  $q = e^{2\pi i/n}$ , such that the algebra generated by  $X$  and  $Y$  is all of  $M_n(\mathbb{C})$ . Because  $X^n$  and  $Y^n$  are reducible by the Cayley-Hamilton Theorem, the set  $P = \{X^i Y^j | 0 \leq i, j \leq n - 1\}$  must span all of  $L_*$ . Since  $M_n(\mathbb{C})$  has dimension  $n^2$  and  $P$  contains  $n^2$  matrices,  $P$  is in fact a basis. Thus the  $(2n - 2)$ -length product  $X^{n-1} Y^{n-1}$  is linearly independent from any products of lesser length, giving us that  $c(\Sigma) = 2n - 2$  for such a set of matrices.

It remains only to show that such matrices do indeed exist. We leave it to the reader to verify that the following matrices satisfy the above conditions.

$$X = \begin{bmatrix} 1 & & & \\ & q & & \\ & & \ddots & \\ & & & q^{n-1} \end{bmatrix}, \quad Y = \begin{bmatrix} & & & 1 \\ & & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

Lower bounds for certain sets of matrices can be obtained. Paz's suggested lemma is set up to deal with sets of matrices for which  $r_*$  could be as great as  $n^2$ . With more information about the dimension of  $L_*$  for a given set of matrices, the conditions of the lemma can be tightened, resulting in lower bounds for  $c(\Sigma)$ , as the following shows.

We prove a slightly more restrictive form of Proposition 2.5, which we then use to prove Theorem 3.2.

**Proposition 3.1.** *Let  $m = 2n - 2$ . If for some  $k \leq 2n - 3$  the corresponding condition from among*

$$\begin{aligned} r_k - r_{k-1} &\leq k \text{ for } 1 \leq k \leq n - 1 \\ r_k - r_{k-1} &\leq 2n - k - 2 \text{ for } n \leq k \leq 2n - 3 \end{aligned}$$

*holds then  $c(\Sigma) \leq m - 1$ .*

*Proof.* Lemma 2.4 tells us that in an ordered word of length  $2n - 1$  there are at least  $N + 2$  formally distinct  $k$ -subwords. Since decreasing to length  $2n - 2$  eliminates at most one of these  $k$ -subwords, there will still be at least  $N + 1$ . Since we only need more than  $N$ , the proof of Proposition 3.1 then follows directly from the proof of Proposition 2.5.  $\square$

**Theorem 3.2.** *Let  $\Sigma$  be as before, with the added restriction that it does not generate all of  $M_n(\mathbf{k})$ . Then  $c(\Sigma) \leq 2n - 3$ .*

*Proof.* Now we proceed as before. If  $c(\Sigma) = m \geq 2n - 2$ , then none of the above conditions can hold. This implies  $r_0 = 1, r_1 - r_0 \geq 2, r_2 - r_1 \geq 3, \dots, r_{n-1} - r_{n-2} \geq n, r_n - r_{n-1} \geq n - 1, \dots, r_{2n-3} - r_{2n-4} \geq 2$ . Then we have  $r_{2n-3} \geq 1 + 2 + \dots + n - 1 + n + n - 1 + \dots + 2 = 2(n(n - 1))/2 + n - 1 = n^2 - 1$ . Because of the restriction placed on the algebra generated by  $\Sigma$ ,  $n^2 - 1 \geq r_*$ . As before, this contradicts  $c(\Sigma) \geq 2n - 2$ .  $\square$

For representations of Lie algebras  $c(\Sigma)$  is bounded by  $2n - 3$  as well. No Lie algebra consisting of two matrices generates all of  $M_n$ , and for a Lie algebra of three or more matrices to do so those three matrices, together with the identity, must be linearly independent, implying that  $r_1 - r_0 \geq 3$ . This allows a proof similar to that given for Theorem 3.2 since

now the sum of the  $r_{i+1} - r_i$  terms will be greater than or equal to  $n^2$ . We leave as an open question whether the bound of  $2n - 3$  is sharp for representations of Lie algebras. We have looked for an example achieving this bound but have been unable to find one.

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