Research Statement

Zajj Daugherty

November 8, 2009

My research is in combinatorics and representation theory. Specifically I use combinatorial techniques to study algebras that are generalizations of the group algebra of the symmetric group, Brauer algebras, and graded Hecke and Brauer algebras.

Combinatorial representation theory uses combinatorial objects (partitions, tableaux, etc.) to understand and control the representation theory of algebraic objects (Lie groups, Lie algebras, reflection groups, etc.). Early work of Frobenius laid the groundwork for studying groups using character theory. He specifically outlined the rich and beautiful structure of the character theory of the symmetric group through the use of combinatorial tools. In 1901, Schur's thesis provided the link between the combinatorics developed for the symmetric group S_k and similar phenomena appearing in the character theory of the general linear group GL_n. He brought these two groups together using the insight that the action of GL_n on tensor space $(\mathbb{C}^n)^{\otimes k}$ centralizes the action of S_k, and that this relationship could be used to produce irreducible modules. This phenomenon is now known as *Schur-Weyl duality*—the general statement is that if A is the full centralizer of the action of a semisimple algebra B on a B-module M, then M decomposes into direct irreducible summands

$$M \cong \bigoplus_{\lambda} A^{\lambda} \otimes B^{\lambda}$$

as an (A, B) bimodule, where A^λ and B^λ are distinct irreducible modules for A and B, respectively. This means that the representation theory of A is in many ways determined by the representation theory of B, and vice versa.

The centralizer property stimulated many advances in the development of *tensor power centralizer algebras*, algebras of operators which preserve symmetries in a tensor space. Striking examples include:

(1) the *Brauer algebras* centralize the action of symplectic and orthogonal groups on tensor space $(\mathbb{C}^n)^{\otimes k}$;
(2) the *graded Hecke algebra of type A* centralizes the action of $\mathfrak{sl}_n$ on $L(\lambda) \otimes (\mathbb{C}^n)^{\otimes k}$, where $L(\lambda)$ is the irreducible $\mathfrak{sl}_n$ module indexed by a partition λ;
(3) the *degenerate affine Wenzl algebra* centralizes the action of symplectic and orthogonal groups on $L(\lambda) \otimes (\mathbb{C}^n)^{\otimes k}$.

A paper of Orellana and Ram [OR] provided a unified approach to studying tensor power centralizer algebras, including the *affine and cyclotomic Hecke and Birman-Murakami-Wenzl algebras*.

Recent work in the study of loop models and spin chains in statistical mechanics uncovered yet another potential use of Schur-Weyl duality (see [GN]). Specifically, a connection was discovered between the two-boundary Temperley-Lieb algebra and a quotient of the affine Hecke algebra of type C. Since the Temperley-Lieb algebra is the centralizer of the quantum group $\mathcal{U}_q\mathfrak{sl}_2$ on tensor space $M \otimes N \otimes (\mathbb{C}^2)^{\otimes k}$, where M and N are simple $\mathcal{U}_q\mathfrak{sl}_2$-modules, this connection opened the community's eyes to the possibility of constructing affine Hecke algebra type C modules explicitly using Schur-Weyl duality tools.

My thesis studies the centralizer of the action of $\mathfrak{g}$ on $M \otimes N \otimes V^{\otimes k}$, where $\mathfrak{g}$ is a finite dimensional complex reductive Lie algebra and M, N, and V are finite dimensional irreducible $\mathfrak{g}$-modules. The new definition is that of the *two-boundary graded braid group* $\mathcal{G}_k$. The structure of $\mathcal{G}_k$ is

$$\mathbb{C}[z_0, z_1, \ldots, z_k] \otimes \mathbb{C}[y_1, \ldots, y_k] \otimes \mathbb{C}[x_1, \ldots, x_k] \otimes \mathbb{C}S_k,$$

with relations twisting the polynomial rings and the group algebra of the symmetric group together. The first main theorem states that $\mathcal{G}_k$ acts on $M \otimes N \otimes V^{\otimes k}$ and that this action commutes with the action of $\mathfrak{g}$. In many examples, these actions produce the full centralizer $\mathrm{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$.

An example which is developed in detail in my thesis is the case where $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{gl}_n$, $M = L((a^p))$ (the finite dimensional $\mathfrak{g}$-module indexed by the rectangular partition with p parts of length a), $N = L((b^q))$, and $V = L((1^1)) \cong \mathbb{C}^n$. With suitable restrictions on a, b, p and q, I prove that the centralizer is the image of the $\mathcal{G}_k$-action on $M \otimes N \otimes V^{\otimes k}$. I define a universal object, the *extended two-boundary graded Hecke algebra* $\mathcal{H}_k^{\mathrm{ext}}$, which is a quotient of $\mathcal{G}_k$ and has these centralizers as quotients. It is a generalization of the graded Hecke algebra and is related to the two-boundary affine Hecke algebra.

I further determine in detail the representation theory of these centralizers and $\mathcal{H}_k^{\mathrm{ext}}$. The irreducible representations are indexed by an appropriate class of partitions, described in detail in my thesis. Using the combinatorics of Young tableaux I describe these representations explicitly by specifying a basis and the action of $\mathcal{H}_k^{\mathrm{ext}}$ on the basis. The basis elements are indexed by paths in a lattice of partitions, and the formulas for the action of $\mathcal{H}_k^{\mathrm{ext}}$ are given in terms of contents of boxes in the partitions.

Future Research

Connections to type C phenomena. One subalgebra of $\mathcal{H}_k^{\mathrm{ext}}$, the *two-boundary graded Hecke algebra* $\mathcal{H}_k$, is strikingly similar to $\mathbb{H}$, the graded Hecke algebra of type C. This observation suggests the possibility of studying representations of type C Hecke algebras using Schur-Weyl duality techniques.

A first step to understanding the similarities between these two algebras is to investigate the center of $\mathcal{H}_k$. The action of central elements is a primary tool in the classification of irreducible representations, and the theory for $\mathbb{H}$ is well developed. A preliminary characterization of the center of $\mathcal{H}_k$ shows it to have similar structure to the center of $\mathbb{H}$—it contains a ring of polynomials symmetric with respect to an action of the type C Weyl group. It remains to be determined whether this subring forms the entire center. We expect that the center will be fully determined through a better understanding of the basis of $\mathcal{H}_k^{\text{ext}}$.

The next step will be to determine the parallels between the representation theory of $\mathbb{H}$ and that of $\mathcal{H}_k$. I have preliminarily compared the Schur-Weyl duality results for $\mathcal{H}_k$ to the previous combinatorial approach via the analysis of intertwining operators in type C (see [Ra]). The calibrated representations of the graded Hecke algebra of type C are indexed by skew shapes which can be recognized in the combinatorial structure of the calibrated representations of the two-boundary graded Hecke algebra. Furthermore, the dimensions of these representations align accordingly in generic cases. With further study of $\mathbb{H}$, I will be able to describe the correspondence between these combinatorial structures in detail.

As a culmination, my goal is to form a precise statement about how we are able to draw valuable information about $\mathbb{H}$ from $\mathcal{H}_k$. In the quantized version, there is an isomorphism between the affine Hecke algebra of type C and the two-boundary Hecke algebra. In [Lu], Lusztig studies the correspondence between the affine algebras and their graded versions. From this study, we expect an analogous isomorphism between the graded Hecke algebra of type C and the two-boundary graded Hecke algebra. In my thesis I give a presentation of $\mathcal{H}_k$ which nearly reveals such an isomorphism. We can also see that the representation theory is strikingly similar. One contribution I may make through this investigation would be to provide an alternate definition of the graded Hecke algebra of type C which clarifies the isomorphism between it and the affine Hecke algebra of type C as explored in [Lu].

Centralizers in type B, C, and D. The *one boundary Brauer algebra* (otherwise known as the *degenerate affine Wenzl algebra*) $\mathcal{B}_k^{(1)}$ was defined by Nazarov in [Na] to capture the action of Jucys-Murphy operators on the irreducible representations of Brauer algebras. A consequence of Nazarov's work is that a quotient of $\mathcal{B}_k^{(1)}$ centralizes the action of the orthogonal and symplectic groups on tensor spaces of the form $M \otimes V^{\otimes k}$, where $M = L(\lambda)$ and $V = L((1^1))$. In work with Ram and Virk, we have defined the (one boundary) graded braid group $\mathcal{G}_k^{(1)}$ and shown that $\mathcal{B}_k^{(1)}$ is a quotient of $\mathcal{G}_k^{(1)}$ (paper in progress: [DRV]).

I would like to do an analogous study of the centralizer of the action of $\mathfrak{g}$ on $M \otimes N \otimes V^{\otimes k}$ when $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$ or $\mathfrak{sp}_{2n}(\mathbb{C})$ (the Lie algebras associated to the orthogonal and symplectic groups). We expect the centralizer of the action of $\mathfrak{g}$ on $M \otimes N \otimes V^{\otimes k}$ to be a quotient of the two-boundary graded braid group $\mathcal{G}_k$ as defined in my thesis. I will define the two-boundary graded Brauer algebra $\mathcal{B}_k$ as the universal algebra which has centralizers of the action of $\mathfrak{g}$ on $M \otimes N \otimes V^{\otimes k}$ as quotients for suitable choices of M and N. We expect this universal object to be a quotient of $\mathcal{G}_k$. The structure of $\mathcal{G}_k$ is largely based on the structure of the three images of $\mathcal{G}_k^{(1)}$ it contains—one corresponding to M, one corresponding to N, and one corresponding to $M \otimes N$. Similarly, $\mathcal{B}_k$ contains three images of $\mathcal{B}_k^{(1)}$. The structure and representation theory of $\mathcal{B}_k^{(1)}$ is described in [AMR]. One hopes that the construction of $\mathcal{B}_k$ can be done by understanding how the three images of $\mathcal{B}_k^{(1)}$ twist together, and combining that with the known structure of $\mathcal{B}_k^{(1)}$. This approach is completely analogous to that of the analysis of the Hecke algebras done in my thesis.

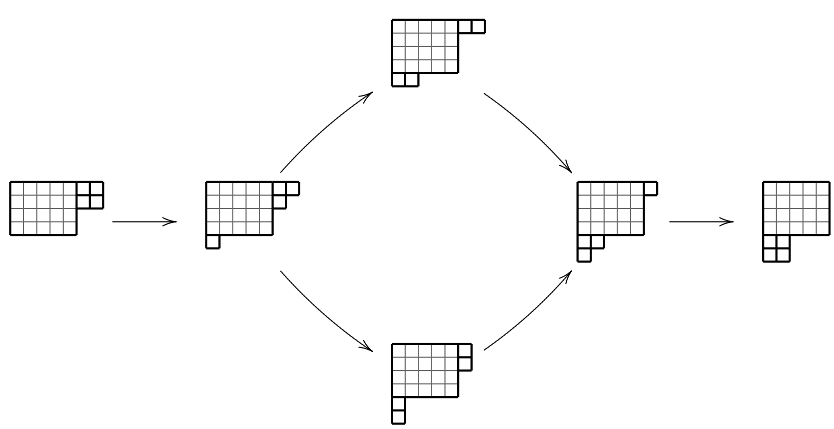
Once the two-boundary graded Brauer algebra is defined, my next goal will be to explicitly construct the irreducible representations of $\mathcal{B}_k$ using combinatorial tools. The decomposition of $M \otimes N$ when M and N are simple modules corresponding to rectangular partitions is made precise in [Ok], so the decomposition of $M \otimes N \otimes V^{\otimes k}$ is well controlled. We expect the irreducible modules of $\mathcal{B}_k$ to be indexed by a class of partitions and to have a basis whose elements are indexed by certain up-down tableaux. Both the class of partitions and of up-down tableaux will be given by the decomposition of $M \otimes N \otimes V^{\otimes k}$. The formulas for the action of $\mathcal{B}_k$ will be in a similar form to the formulas for the action of $\mathcal{H}_k^{\text{ext}}$ given in my thesis work and the formulas for the action of $\mathcal{B}_k^{(1)}$ in [AMR]. This approach also parallels the analysis of the quantized version of $\mathcal{B}_k^{(1)}$ done in [OR].

Next, I will study of the center of $\mathcal{B}_k$. In work with Ram and Virk ([DRV]), we thoroughly study the center of the one boundary graded Brauer algebra $\mathcal{B}_k^{(1)}$, and its quantized version, the affine Birman-Murakami-Wenzl algebra. Each of these algebras is defined with a choice of an infinite family of parameters awkwardly subject to admissibility conditions. By studying the centers of $\mathcal{B}_k^{(1)}$ and the affine BMW algebra, we define two algebras which do not depend on a choice of parameters, but which specialize to $\mathcal{B}_k^{(1)}$ and the affine BMW algebra, respectively. A priori, $\mathcal{B}_k$ should depend on a *pair* of infinite families of parameters subject to admissibility conditions. This leads to the question: Is there an algebra which does not depend on a choice of parameters, but which specializes to $\mathcal{B}_k$ for any admissible choice of parameters? A proper study of the center of $\mathcal{B}_k$ should lead to the definition of such an algebra.

Non-calibrated representations. In a paper of Orellana and Ram [OR], they construct a functor from quantum group $\mathcal{U}_q\mathfrak{g}$ modules to affine braid group modules. This functor takes finite dimensional $\mathcal{U}_q\mathfrak{g}$-modules to "calibrated" modules, Verma modules to "standard" modules, and irreducible modules to irreducible modules (under appropriate conditions). The affine and cyclotomic Hecke and Birman-Murakami-Wenzl modules are then described as quotients of the affine braid group. They recover representations of these algebras by considering the cases when $\mathfrak{g}$ is type $\mathfrak{sl}_n$, $\mathfrak{so}_n$ and $\mathfrak{sp}_{2n}$. In my thesis, I construct the calibrated modules for $\mathcal{H}_k^{\text{ext}}$ by mapping $\mathcal{H}_k^{\text{ext}}$ into $\text{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$, where M, N, and V are finite dimensional. I hope to construct a similar functor from the set of pairs of $\mathfrak{g}$-modules M and N to irreducible $\mathcal{G}_k$-modules. This will allow me to better understand the representation theory of $\mathcal{G}_k$, $\mathcal{H}_k^{\text{ext}}$, and $\mathcal{B}_k$. I should then be able to describe the combinatorial structure of not only calibrated modules for these three algebras, but standard modules as well. The results should parallel the formulas given in [OR].

Full centralizers. Under certain conditions on the relative sizes of M and N, I have found that the image of $\mathcal{H}_k^{\text{ext}}$ in $\text{End}_{\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$ recovers the entire centralizer. The fact that this happens as often as it does is remarkable. The beauty of the representation described in my thesis work is that, when it surjects, it provides a construction of the full centralizer that relies on the use of very simple operators. When the representation is not surjective, we can see through an analysis of the decomposition of $M \otimes N \otimes V^{\otimes k}$ that the image of $\mathcal{H}_k^{\text{ext}}$ differs from the full centralizer only by a commutative subalgebra—a portion of image of the center of $\mathfrak{g}$ acting on $M \otimes N$ not contained in the image of $\mathcal{H}_k^{\text{ext}}$.

In my treatment of the decomposition of $M \otimes N$, I describe a partial order on $\mathcal{T}_{M,N}$, the partitions indexing nonzero components of $M \otimes N$. This partial order is compatible with the dominance order and captures an inductive process for constructing each partition in $\mathcal{T}_{M,N}$ by moving successive boxes. For example, if $M = L((5^4))$ and $N = L((2^2))$, then $M \otimes N$ has six components. The partitions indexing these six components and the partial order on this set is illustrated in the diagram below.



I hope to use this inductive process to construct and control projections onto each component of $M \otimes N$, thus calculating the full centralizer of the $\mathfrak{g}$-action on $M \otimes N \otimes V^{\otimes k}$.

In the study of the two-boundary Brauer algebra, we expect similar phenomena. I hope to harness similar tools to extend quotients of $\mathcal{B}_k$ to full centralizers in non-generic cases.

References

[AMR] S. Ariki, A. Mathas and H. Rui, *Cyclotomic Nazarov-Wenzl algebras*, Nagoya Math. J. **182** (2006), 47–134. MR2235339

[AS] T. Arakawa and T. Suzuki, *Duality between $\mathfrak{sl}_n(\mathbb{C})$ and the Degenerate Affine Hecke Algebra*, J. Algebra **209** (1998), 288–304.

[Br] R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Ann. of Math. **38** (1937), 857872.

[DRV] Z. Daugherty, A. Ram, R. Virk, *Affine and graded BMW algebras*, In progress.

[GN] J. de Gier and A. Nichols, *The two-boundary Temperley-Lieb algebra*, J. Algebra **321** (2009) 11321167.

[Lu] G. Lusztig, *Cuspidal local systems and graded Hecke algebras I*, Inst. Hautes Etudes Sci. Publ. Math. **67** (1988), 145202.

[Na] M. Nazarov, *Youngs Orthogonal Form for Brauers Centralizer Algebra*, J. Algebra **182** (1996), 664–693.

[Ok] S. Okada, *Applications of Minor Summation Formulas to Rectangular-Shaped Representations of Classical Groups*, J. Algebra **205** (1998), 337–367.

[OR] R. Orellana and A. Ram, *Affine braids, Markov traces and the category $\mathcal{O}$*, in *Proceedings of the International Colloquium on Algebraic Groups and Homogeneous Spaces Mumbai 2004*, V.B. Mehta ed., Tata Institute of Fundamental Research, Narosa Publishing House, Amer. Math. Soc. (2007) 423-473.

[Ra] *Affine Hecke algebras and generalized standard Young tableaux*, J. Algebra, **230** (2003), 367–415.