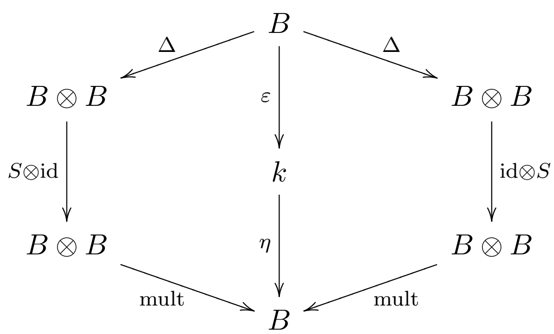
Hopf Algebras

Zajj Daugherty

Big ideas: A *bialgebra* B over a field k is a structure which is both a *unital* (has identity) associative algebra and a *coalgebra* (has a map $B \to B \times B$) over k, such that these structures are compatible. Compatibility just means that the *comultiplication* and the *counit* ε are both unital algebra homomorphisms, or equivalently, that the multiplication and the unit of the algebra both be coalgebra morphisms (these are equivalent in that they are expressed by the same diagram).

A *Hopf algebra* is a bialgebra with an *anitpode*—a map which generalizes the inversion map on a group which sends g to g^{-1} (if an antipode exists, it must be unique)—such that



commutes, where $\eta : k \to B$ is the unit map $\eta(c) = c \cdot 1$.

Example: group algebras A super-simple example of these are group algebras with normal multiplication, comultiplication given by

$$g \overset{\Delta}{\longmapsto} g \otimes_k g,$$

and antipode

$$g \overset{S}{\longmapsto} g^{-1}.$$

Example: enveloping algebras Let $\mathfrak{g}$ be a Lie algebra over k. The enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ is the algebra generated by $\mathfrak{g}$ with the relations $xy - yx = [x, y]$ for all $x, y \in \mathfrak{g}$. If V is a $\mathfrak{g}$-module (i.e. a $U\mathfrak{g}$-module), then this action extends to $V \otimes V$ by $x \cdot (v_1 \otimes v_2) = xv_1 \otimes v_2 + v_1 \otimes xv_2$, for all $x \in \mathfrak{g}$, $v_1, v_2 \in V$. This reflects the Hopf algebra structure on $U\mathfrak{g}$ given by

$$\begin{aligned} \textit{counit} \quad & x \overset{\varepsilon}{\longmapsto} 0, \\ \textit{comultiplication} \quad & x \overset{\Delta}{\longmapsto} x \otimes 1 + 1 \otimes x, \text{ and} \\ \textit{antipode} \quad & x \overset{S}{\longmapsto} -x, \end{aligned}$$

for $x \in \mathfrak{g}$.

Example: polynomial functions on a Lie group Let $B = \mathcal{O}(G)$ be the algebra of complex-valued polynomial functions on a complex Lie group G, and identify $\mathcal{O}(G \times G)$ with $B \otimes B$. Then, B is a Hopf algebra with

$$\begin{array}{rl} \textit{counit} & \varepsilon(f) = f(1), \\ \textit{comultiplication} & (\Delta(f))(g_1, g_2) = f(g_1 g_2), \text{ and} \\ \textit{antipode} & (S(f))(g) = f(g^{-1}). \end{array}$$

If $\mathfrak{g}$ is the Lie algebra of G, then the above two examples are dual to one another: Define a bilinear form $\mathcal{O}(G) \otimes U\mathfrak{g} \to \mathbb{C}$ by $\langle f, x \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f(\exp(tx))$. Then,

$$\begin{aligned} \langle fg, x \rangle &= \langle f \otimes g, \Delta(x) \rangle, \\ \langle 1, x \rangle &= \varepsilon(x), \\ \langle \Delta(f), x \otimes y \rangle &= \langle f, xy \rangle, \\ \varepsilon(f) &= \langle f, e \rangle, \\ \langle S(f), x \rangle &= \langle f, S(x) \rangle, \\ \langle f^*, x \rangle &= \overline{\langle f, S(x)^* \rangle}. \end{aligned}$$

Why topologists care. Let G be a topological group. There are the two familiar maps

$$G \times G \xrightarrow{\nabla} G \quad \text{(multiplication)} \qquad \text{and} \qquad G \xrightarrow{\Delta} G \times G \quad \text{(the diagonal in } G\text{)}.$$

Thus a functor

$$F : \{\text{top spaces}\} \to \{\text{vector spaces}\}$$

which takes cross products into tensor products gives me a vector space $F(G)$ and maps

$$F(G) \otimes F(G) \to F(G) \qquad \text{and} \qquad F(G) \to F(G) \otimes F(G).$$

So if functors like F let us in some sense see the structure of topological spaces reflected in the category of vector spaces, then what they turn groups into Hopf algebras. If F "linearizes" the category of topological spaces somehow, then this says that "the linearization of a group is Hopf algebra."