2D EULER EQUATION ON THE STRIP: STABILITY OF A RECTANGULAR PATCH

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Abstract. We consider the 2D Euler equation of incompressible fluids on a strip $\mathbb{R}\times\mathbb{T}$ and prove the stability of the rectangular stationary state $\chi_{|x|<L}$ for large enough L.

1. Introduction

In this paper we will consider the stability of a certain class of steady solutions to the Euler equation in a two dimensional cylindrical domain. The study of such stability questions is well developed in the planar case. In the plane, the primary focus has been in the stability of circular patches, starting with [8], which studied the evolution of a circular patch in a bounded domain and proved stability using a spectral argument. In a similar vein, [5] used conserved quantities to derive a bound on the diameter growth. The use of conserved quantities was also key in [7]. There has also been work studying the stability and instability of other steady solutions in the plane, such as the Kirchhoff ellipse cf. [4].

In other domains, these types of questions are less well understood. In the strip $\mathbb{R}\times[0,a]$, the work of Caprino and Marchioro [3] shows the stability of monotonically increasing steady vorticity distributions with restrictive conditions on the associated velocity. More recently, Bedrossian and Masmoudi [2] showed nonlinear stability of Couette flow in the cylinder $S=\mathbb{R}\times\mathbb{T}$.

This paper considers steady patch solutions of the form $\chi_{E_0}(z)$ where $E_0=[-L,L]\times\mathbb{T}$ to the problem

$$\partial_t\theta+\nabla\theta\cdot u=0,\quad \theta|_{t=0}=\theta_0=\chi_E \tag{1}$$

for a compact set $E\subset S$. The velocity $u(z,t)$ is related to the vorticity θ via a cylindrical Biot-Savart law. Let the stream function Ψ be the function that solves the elliptic problem

$$(2\pi)^{-1}\Delta\Psi=\theta,\quad \lim_{x\to+\infty}\partial_1\Psi(x,y,t)=-\lim_{x\to-\infty}\partial_1\Psi(x,y,t),\quad |\Psi(x,y,t)|\leqslant C(|x|+1). \tag{2}$$

The cylindrical Biot-Savart law is then

$$u=\nabla^\perp\Psi=k*\theta,\quad \Gamma(x,y)=\frac{1}{2}\log(\cosh(x)-\cos(y)),\quad k(x,y)=\nabla^\perp\Gamma(x,y)=\frac{(-\sin(y),\sinh(x))}{2(\cosh(x)-\cos(y))}. \tag{3}$$

The velocity u defined by the cylindrical Biot-Savart law exists and is unique by the following Lemma.

Lemma 1.1. *Given compactly supported $\theta(z,t)\in L^\infty(S)$, all solutions to equation (2) are given by*

$$\Psi(x,y)=\Gamma*\theta+C$$

for some constant C.

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Proof. To see uniqueness, let Ψ_1 and Ψ_2 be two solutions to the elliptic problem (2). Then $\widetilde{\Psi} = \Psi_1 - \Psi_2$ solves

$$\Delta\widetilde{\Psi} = 0, \quad \lim_{x\to+\infty} \partial_1\widetilde{\Psi}(x,y,t) = -\lim_{x\to-\infty} \partial_1\widetilde{\Psi}(x,y,t).$$

The upper bound on Ψ_1 and Ψ_2 guarantees that the $\widetilde{\Psi}$ grows at most linearly. If we extend $\widetilde{\Psi}$ to all of $\mathbb{R}^2$ by periodicity in y, Liouville's Theorem for harmonic functions gives that $\widetilde{\Psi} = C_1x + C_2y + C_3$. Since $\widetilde{\Psi}$ is periodic in y, $C_2 = 0$. The conditions on $\partial_1\widetilde{\Psi}$ mean that $C_1 = 0$. Thus $\widetilde{\Psi} = C$.

It remains to show that $\Gamma * \theta$ is a solution to (2). One can directly check that $\Delta\Gamma = 0$, $z \neq 0$. Moreover,

$$(2\pi)^{-1}\Gamma(z) = \frac{1}{\pi}\log|z| + C + o(1), \quad |z| \to 0$$

and $(2\pi)^{-1}\Delta(\Gamma * \theta) = \theta$. □

In contrast to the [3], these boundary conditions give a counter rotating velocity as $|x| \to \infty$ with a linear transition within the patch E_0. Additionally, this Biot-Savart law does not produce velocity in $L^2(S)$. The kernel $k(x,y)$ can be decomposed into two pieces, one in $L^1(S)$ and the other which is bounded. Observe that

$$u(x,y,t) = \int_S \frac{\big((-\sin(y-\xi_2), \operatorname{sgn}(x-\xi_1)(\cos(y-\xi_2) - e^{-|x-\xi_1|})\big)}{2(\cosh(x-\xi_1) - \cos(y-\xi_2))}\theta(\xi)d\xi + \frac{1}{2}\int_S (0, \operatorname{sgn}(x-\xi_1))\theta(\xi)d\xi. \tag{4}$$

Notice that the first term is convolution with a kernel in $L^1(S)$, as near 0, it behaves like $(x^2+y^2)^{-1/2}$ and away from 0 it decays exponentially fast. The second term, however, does not decay at all so the total kinetic energy of this problem could be infinite. However, we can define the regularized energy $F(\theta)$ for the equation (1) and compactly supported $\theta(z,t) \in L^\infty(S)$ in the following way:

$$F(\theta)(t) = \int_S\int_S \theta(z,t)\theta(\xi,t)\log(\cosh(x_1-x_2) - \cos(y_1-y_2))dzd\xi, z = (x_1,y_1), \xi = (x_2,y_2). \tag{5}$$

When we consider the evolution of a patch under this flow, we will show in section 2 that the total mass, the first coordinate of center of mass, and regularized energy are conserved.

We will use the following notation in the paper. If A and B are sets, $A\Delta B$ stands for the symmetric difference. The symbol C will denote an absolute constant, and its actual value can change from formula to formula. If $f_{1(2)}(x)$ are two positive functions and

$$\sup_x \frac{f_1(x)}{f_2(x)} < \infty$$

we will write $f_1 \lesssim f_2$. This is equivalent to writing $f_1 = O(f_2)$.

We can now state our main Theorem. Similar to [7], we will show that the steady patch solution $E_0 = [-L, L] \times \mathbb{T}$ for sufficiently large L is stable for all times. It is convenient for our calculations to introduce what we will call a point of centering.

Definition 1.1. A *point of centering* $x_c(t)$ for a patch $E(t)$ is the value in $\mathbb{R}$ so that

$$\int_{[x_c,\infty)\times\mathbb{T}} \chi_{E(t)}dz = \int_{(-\infty,x_c]\times\mathbb{T}} \chi_{E(t)}dz.$$

Notice that this point is not necessarily unique and the set of all such points is always a segment or a single point. We use a point of centering to make the comparison between the evolved patch $E(t)$ and the simple rectangle E_0 more natural.

Our main result is the following Theorem.

Theorem 1.1. *There is an absolute constant $L_0 > 2$ such that the following statement is true. If*

(a) $L > L_0$, $\epsilon < 1$,

(b) *E is a compact subset of S and 0 is one of its points of centering,*
(c) $|E| = 4L\pi$,
(d) *the regularized energy satisfies*

$$F(\chi_E) = F(\chi_{E_0}) + O(L\epsilon^2), \tag{6}$$

(e) *function θ solves 2D Euler equation (1) with the Biot-Savart law given by (2) and (3), then $\theta(t) = \chi_{E(t)}$ and $E(t)$ satisfies*

$$\int_S ||x - x_c(t)| - L|\chi_{E(t)\,\Delta\, E_0(t)} dxdy \lesssim \epsilon^2, \tag{7}$$

$$|x_c(t)| \lesssim L^{-1}\epsilon^2 \tag{8}$$

for all $t > 0$. Above $x_c(t)$ is any point of centering for $E(t)$ and $E_0(t) = [x_c(t) - L, x_c(t) + L] \times \mathbb{T}$.

Moreover, if $\mu > \epsilon$, then

$$|(E(t)\,\Delta\, E_0(t)) \cap \{||x - x_c(t)| - L| > \mu\}| \lesssim \epsilon^2\mu^{-1}.$$

This result has a similar structure to the result in [7] for circular patches in the plane, but with a point of centering $x_c(t)$ in the role of the center of mass. However, that proof relies on conserved quantities that do not hold in the cylindrical case, namely $\int_{\mathbb{R}^2} |z|^2\theta(z)dz$. Instead, our argument uses the one dimensional nature of the cylindrical problem and the conservation of regularized energy. In the next section, we will establish the necessary conserved quantities. In the third section, we prove the main result on the stability.

2. Preliminaries

To proceed with our arguments on stability, we need a result equivalent to Yudovič's result for the evolution of $L^1 \bigcap L^\infty$ solution [9, 2]. We are working on an unbounded domain with periodicity in one direction. If we consider the periodic extension of our problem to $\mathbb{R}^2$, we are interested in bounded solutions on $\mathbb{R}^2$ with some decay in one direction. Recent work by Kelliher and collaborators gives existence and uniqueness for solutions to the Euler equations on $\mathbb{R}^2$ for velocity u and associated vorticity $\theta = \nabla \times u$ (defined in the sense of distributions) both bounded, with no decay requirement. These results also include an adaptation of the standard Biot-Savart law on $\mathbb{R}^2$ to relate u and θ despite the lack of convergence of the standard integral identities.

We will apply this work in Appendix A to show the following Theorem:

Theorem 2.1. *Let $\theta_0(z)$ for $z \in S = \mathbb{R} \times \mathbb{T}$ be in $L^\infty(S)$ with compact support. Then there exists unique (u, θ) with $u \in L^\infty(S)$ and $\theta \in L^\infty(S)$ with compact support such that $\partial_t\theta + u \cdot \nabla\theta = 0$ in the sense of distributions with*

$$u(z,t) = \nabla^\perp(\Gamma * \theta) = \int_S \frac{(-\sin(y - \xi_2), \sinh(x - \xi_1))}{2(\cosh(x - \xi_1) - \cos(y - \xi_2))}\theta(\xi, t)d\xi$$

and $\theta(z, 0) = \theta_0(z)$.

We postpone the proof of this Theorem until the Appendix.

Once we have established existence and uniqueness, we can study the conserved quantities of the equation.

Proposition 2.1. *For $\theta(z,t)$ a solution to (1), the following quantities are conserved:*
(1) *the total mass $M = \int_S \theta(z)dz$,*
(2) *the horizontal center of mass $x_0 = \int_S x\theta(z)dz$,*
(3) *the total energy $F(\theta) = 2\int_S\int_S \theta(z)\theta(\xi)\Gamma(z - \xi)d\xi dz$.*

Proof. The arguments for Theorem 2.1 include that the vorticity θ is transported by the flow. Therefore, conservation of mass follows immediately.

Since we have $\partial_t \theta + u \cdot \nabla \theta = 0$ only in the sense of distributions, we need to show carefully that we have conservation of center of mass and regularized energy. Observe that for a smooth function $\varphi \in C^\infty([0,T], C_0^\infty(S))$, we have the following representation:

$$\int_S \varphi(z,T)\theta(z,T)dz - \int_S \varphi(z,0)\theta(z,0)dz = \int_0^T \int_S \left(\partial_t \varphi + u \cdot \nabla \varphi\right) \theta(z,t) dz dt.$$

To show our other conserved quantities, we need to choose our smooth bump function so that the desired quantity appears on the right hand side of the expression above.

From (4), we can bound the velocity u by

$$\|u(t)\|_{L^\infty(S)} \leqslant \|k_1\|_{L^1(S)} \|\theta(t)\|_{L^\infty(S)} + \|\theta(t)\|_{L^1(S)},$$

where $k_1(z) = k(z) - (0, \operatorname{sgn}(x))$. The L^1 bound on k is independent of time, and both $\|\theta\|_{L^1(S)}$ and $\|\theta\|_{L^\infty(S)}$ are conserved in time. Therefore,

$$\|u(t)\|_{L^\infty(S)} \lesssim \|\theta(0)\|_{L^\infty(S)} + \|\theta(0)\|_{L^1(S)}, \tag{9}$$

and we know that up to a time T, θ is compactly supported. Let $b(x)$ be a smooth bump in the x-direction so that $b \equiv 1$ for every x in the set $[-R,R]$ where R is chosen so that $\operatorname{supp}(\theta(z,t)) \subseteq [-R,R]$ for all $t \in [0,T]$.

To see conservation of the center of mass, let $\varphi(z,t) = xb(x)$. Then,

$$\begin{aligned}
\int_S x\theta(z,T)dz - \int_S x\theta(z,0)dz &= \int_0^T \int_S \left(\partial_t \varphi + u \cdot \nabla \varphi\right) \theta(z,t) dz dt \\
&= \int_0^T \int_S u_1(z,t) \partial_1 (xb(x)) \theta(z,t) dz dt \\
&= \int_0^T \int_S u_1(z,t)\theta(z,t) dz dt + \int_0^T \int_S x u_1(z,t) b'(x) \theta(z,t) dz dt.
\end{aligned}$$

The second term is clearly 0, as $b'(x) = 0$ on the support of $\theta(z,t)$. The first term can be rewritten as

$$\int_0^T \int_S \int_S \frac{-\sin(y_1 - y_2)}{\cosh(x_1 - x_2) - \cos(y_1 - y_2)} \theta(\xi,t)\theta(z,t) d\xi dz$$

which is also 0, as the kernel is odd and rapidly decaying.

To see the conservation of regularized energy, we repeat a similar argument with $\varphi(z,t) = b(x)\Psi_\varepsilon(z,t)$ where

$$\Psi_\varepsilon(z,t) = \rho_\varepsilon * \int_S \log(\cosh(x - \xi_1) - \cos(y - \xi_2))(\rho_\varepsilon * \theta) d\xi.$$

for a smooth, compactly supported bump $\rho_\varepsilon(z,t)$ on $[0,T] \times S$ defined as follows. Let $r(x)$ be a smooth bump supported in $[-1,1]$ on $\mathbb{R}$ with $\int r = 1$. Let $r_\varepsilon(x) = \varepsilon^{-1} r(x/\varepsilon)$, let $r_{1,\varepsilon}(y)$ be the periodic extension of r_ε on $[-\pi,\pi]$, and let $r_{2,\varepsilon}(t) = r_\varepsilon(t)$. Then, $\rho_\varepsilon(x,y,t) = r_\varepsilon(x) r_{1,\varepsilon}(y) r_{2,\varepsilon}(t)$. The calculations involve changing the order of integration to rearrange the convolution with the mollifier ρ_ε but are otherwise straightforward. □

3. Main Results

We recall that $S = \mathbb{R} \times \mathbb{T}$. We first consider a one-dimensional variational problem which will be important later. Suppose J is a measurable subset of $\mathbb{R}$ and $|J| = 2L$. Assume that J is centered around the origin, such that $|J^+| = L$, $J^+ = J \cap [0,\infty)$. (Notice here again that the "centering points" for any set form a closed interval, which can degenerate to a point). Consider a functional

$$\Phi(\chi_J) = \int_J \int_J |x_1 - x_2| dx_1 dx_2.$$

The following elementary Lemma holds true.

Lemma 3.1. *We have*

$$\Phi(\chi_J) \geqslant \Phi(\chi_{J_0}) + CL \int_{J \,\Delta\, J_0} ||x| - L| dx,$$

where $J_0 = [-L, L]$.

Proof. Notice that this estimate is scale-invariant in L and the actual value of L is not important. It is sufficient to assume that $J^+ = \cup_{j=1}^n I_j$ where I_j are disjoint intervals (placed in the order from left to right). Denote the gaps between them by $\{Q_j\}$ so we have

$$\mathbb{R}^+ = Q_1 \cup I_1 \cup Q_2 \cup I_2 \cup \ldots Q_n \cup I_n \cup [a, \infty).$$

We can allow some gaps to be empty if necessary. The proof will proceed as follows. We will close all gaps $\{Q_j\}$ and estimate the total change in Φ that we denote $\delta\Phi$.

Let $J^{(1)}$ be the configuration obtained by closing the Q_n gap (sliding I_n to the left) and denote the moved interval by $I_n^{(1)}$ (e.g., $|I_n^{(1)}| = |I_n|$, I_{n-1} and $I_n^{(1)}$ are adjacent to each other in $J^{(1)}$). Consider $J' = J \backslash I_n = J^{(1)} \backslash I_n^{(1)}$. We have

$$\Phi(\chi_J) = \Phi(\chi_{J'}) + \Phi(\chi_{I_n}) + 2\int_{J'} d\xi \int_{I_n} (x - \xi) dx, \quad \Phi(\chi_{J^{(1)}}) = \Phi(\chi_{J'}) + \Phi(\chi_{I_n^{(1)}}) + 2\int_{J'} d\xi \int_{I_n^{(1)}} (x - \xi) dx$$

and

$$\Phi(\chi_J) - \Phi(\chi_{J^{(1)}}) = 2|Q_n| \int_{I_n} dx \int_{J'} d\xi \geqslant 2L|Q_n||I_n|,$$

where the last inequality follows from $|J'| \geqslant |J^-| = L$, $J^- = J \cap (-\infty, 0]$.

Compute inductively the total change $\delta_1\Phi$ in Φ obtained by closing all gaps $\{Q_j\}, j = k, \ldots, n$ to the right of L (we close them in the following order: $Q_n, Q_{n-1}, \ldots, Q_k$):

$$\delta_1\Phi \geqslant 2L(|I_n||Q_n| + (|I_n| + |I_{n-1}|)|Q_{n-1}| + \ldots + (|I_n| + \ldots + |I_k|)|Q_k|) =$$

$$2L(|I_n|(|Q_n| + \ldots + |Q_k|) + |I_{n-1}|(|Q_{n-1}| + \ldots + |Q_k|) + \ldots + |I_k||Q_k|). \tag{10}$$

Consider $[L, \infty) \cap J$ and denote $\epsilon = |[L, \infty) \cap J|$. Let us divide all intervals $I_k, \ldots, I_n$ into two groups: those that belong to the interval $[L, L + 2\epsilon]$: $\{I_k, \ldots, I_{j-1}\}$ and those that are to the right of $L + 2\epsilon$: $\{I_j, \ldots, I_n\}$. If $L + 2\epsilon$ is an interior point of some interval, we split this interval into two by creating an empty gap at point $L + 2\epsilon$. Notice that if $I_l \subset [L + 2\epsilon, \infty)$ (i.e., I_l is in the second group), then

$$|Q_l| + \ldots + |Q_k| \geqslant \text{dist}(I_l, L) - \epsilon \geqslant \frac{\text{dist}(I_l, L)}{2}.$$

Therefore, the contribution to (10) coming from the second group of intervals is bounded below by

$$L \sum_{l=j}^n \text{dist}(I_l, L)|I_l| \geqslant 0.5L \int_{x > L + 2\epsilon} (x - L)\chi_J dx.$$

$\Big($Indeed, to see that last inequality we write $I_l = [a_l, b_l]$ and notice that

$$\mathrm{dist}(I_l, L) = a_l - L = (b_l - L) - (b_l - a_l) \geqslant b_l - L - \epsilon \geqslant (b_l - L)/2.\Big)$$

Thus,

$$\delta_1\Phi \gtrsim L \int_{x>L+2\epsilon} (x-L)\chi_J dx = L \int_{x>L+2\epsilon} (x-L)\chi_{J\Delta J_0} dx.$$

Now that all gaps to the right of L are closed, we call the new configuration $\widehat{J}$, intervals are $\{\widehat{I}_j\}$, and the gaps are $\{\widehat{Q}_j\}, j = 1, \ldots m$. For the rightmost interval $\widehat{I}_m$ we have $\widehat{I}_m = [L, L+\epsilon]$ per our construction and thus

$$|\widehat{I}_m| = \epsilon. \tag{11}$$

We are now closing all gaps in $[0, \infty)$ and estimating $\delta_2\Phi$, the change of Φ, from below. Notice first that

$$\sum_{j=1}^{m} |\widehat{Q}_j| = \epsilon \tag{12}$$

and

$$\delta_2\Phi \geqslant 2L(|\widehat{Q}_1|(|\widehat{I}_1| + \ldots + |\widehat{I}_m|) + \ldots + |\widehat{Q}_m| \cdot |\widehat{I}_m|). \tag{13}$$

Observe that $(\widehat{J}\,\Delta\, J_0) \cap [0, L]$ is the union of the $\{\widehat{Q}_l\}$. We now split all gaps $\{\widehat{Q}_l\}$ into two groups: those that belong to $[0, L-2\epsilon]$ (it could be empty if, e.g., $\epsilon > L/2$) and all others. Notice that for each gap $\widehat{Q}_p$ in the first group, we have

$$\sum_{j=p}^{m-1} |\widehat{I}_j| \geqslant \mathrm{dist}(\widehat{Q}_p, L) - \epsilon \geqslant \frac{\mathrm{dist}(\widehat{Q}_p, L)}{2}.$$

Therefore, the contribution to (13) coming from the first group of gaps is at least

$$0.5L \int_{0<x<L-2\epsilon} (L-x)\chi_{J\,\Delta\, J_0} dx.$$

Collecting separately the terms in the right hand side of (13) that contain $\widehat{I}_m$, we get

$$2L|\widehat{I}_m| \cdot (|\widehat{Q}_1| + \ldots + |\widehat{Q}_m|) \geqslant 2L\epsilon^2$$

by (11) and (12). Keeping only the gaps in the first group gives us

$$\delta_2\Phi \geq \left(\sum_{\text{first group of gaps}} 2L|\widehat{Q}_p| \sum_{j=p}^{m-1} |\widehat{I}_j| \right) + 2L|\widehat{I}_m| \cdot (|\widehat{Q}_1| + \ldots + |\widehat{Q}_m|) \gtrsim L \int_{x>0, 0<x<L-2\epsilon} |x-L|\chi_{J\Delta J_0} dx + L\epsilon^2.$$

We combine now the obtained inequalities to estimate the total variation $\delta\Phi = \delta_1\Phi + \delta_2\Phi$ as

$$\delta\Phi \gtrsim L \int_{x>0, |x-L|>2\epsilon} |x-L|\chi_{J\,\Delta\, J_0} dx + L\epsilon^2 \gtrsim L \int_{x>0} |x-L|\chi_{J\,\Delta\, J_0} dx.$$

Arguing in the same way for the half-line $\mathbb{R}^-$, we get the statement of the Lemma since the resulting configuration after closing all gaps is J_0. $\square$

Our first goal is to control the regularized energy functional associated to the Euler equation on S defined in (5). Given an arbitrary vortex patch E, we will need to transition to the vertical average of the patch to control a portion of the energy. To that end, we define following functional

$$\Phi(\rho) = \int_{\mathbb{R}} \int_{\mathbb{R}} |x_1 - x_2| \rho(x_1)\rho(x_2) dx_1 dx_2. \tag{14}$$

Notice first that $\Phi(\rho)<\infty$ implies

$$\int_{\mathbb{R}} |x|\rho(x)dx<\infty.$$

Assume that $\rho_j^+,\rho_j^-\in[0,1], j=0,1,\dots$ are chosen such that

$$\sum_{j=0}^{\infty}\rho_j^{\pm}=1,\quad \sum_{j=1}^{\infty} j\rho_j^{\pm}<\infty.$$

Fix $\delta>0$ and consider the following convex set: $\mathcal{O}$ is the set of functions ρ, defined on $\mathbb{R}$, measurable, and such that $0\leqslant\rho\leqslant 1$ and

$$\int_{j\delta}^{(j+1)\delta}\rho(x)dx=\rho_j^+,\quad \int_{-(j+1)\delta}^{-j\delta}\rho(x)dx=\rho_j^-,\quad j=0,1,\dots$$

In the next Lemma, we will study the following variational problem

$$\inf_{\rho\in\mathcal{O}}\Phi(\rho). \tag{15}$$

In the Lemma B.1 from the Appendix, we prove that a minimizer ρ^* exists.

Lemma 3.2. *If ρ^* is a minimizer then ρ^* is a characteristic function.*

Proof. Notice that if ρ_0 and ρ_1 belong to $\mathcal{O}$, then $\rho_t=t\rho_1+(1-t)\rho_0\in\mathcal{O}$, $t\in(0,1)$ and

$$\Phi''(\rho_t)=\int\int|x_1-x_2|\delta(x_1)\delta(x_2)dx_1dx_2,\quad \delta=\rho_1-\rho_0. \tag{16}$$

Going on the Fourier side, we have

$$\Phi''=-\int|\widehat{\delta}(k)|^2\frac{dk}{2k^2}<0\,, \tag{17}$$

where the last integral makes sense since $\int_{\mathbb{R}}\delta(x)dx=0, \int_{\mathbb{R}}|x||\delta(x)|dx<\infty$ and so $\widehat{\delta}(0)=0, (\widehat{\delta})'\in L^\infty(\mathbb{R})$. That shows concavity of the function in t. Now suppose that ρ^* is not a characteristic function, e.g., there is $\Sigma\subset I_j$ for some I_j, such that $\epsilon<\rho^*<1-\epsilon$ on Σ and $|\Sigma|>0$ for some positive ϵ. Then one can find measurable function ν supported on Σ such that $\int_{\mathbb{R}}\nu dx=0$ and $\nu_t=\rho^*+t\nu\in\mathcal{O}$ for $t\in(-\delta_1,\delta_1)$ with some small positive δ_1. However, the function $\Phi(\nu_t)$ is concave and $t=0$ can not be a local minimum. $\square$

Two previous Lemmas imply

Lemma 3.3. *If ρ is measurable, $0\leqslant\rho\leqslant1$, and*

$$\int_0^\infty\rho dx=L,\int_{-\infty}^0\rho dx=L,$$

then

$$\Phi(\rho)\geqslant\Phi(\chi_{J_0})+CL\int_{\mathbb{R}}||x|-L|\cdot|\rho(x)-\chi_{J_0}|dx. \tag{18}$$

Proof. Indeed, consider the grid $\{j\delta\}, j\in\mathbb{Z}$ with step δ and let

$$\rho_j^+=\int_{j\delta}^{(j+1)\delta}\rho dx,\quad \rho_j^-=\int_{-(j+1)\delta}^{-j\delta}\rho dx,\quad j=0,1,\dots$$

The variational argument given above shows that the value of Φ will decrease if we replace ρ by a minimizer which needs to be a characteristic function χ_J. By Lemma 3.1, we get

$$\Phi(\rho)\geqslant\Phi(\chi_J)\geqslant\Phi(\chi_{J_0})+CL\int_{J\,\Delta\,J_0}||x|-L|dx=\Phi(\chi_{J_0})+CL\int_{\mathbb{R}}||x|-L|\cdot|\chi_J-\chi_{J_0}|dx.$$

Sending $\delta \to 0$ and using

$$\lim_{\delta\to 0}\int_{\mathbb{R}} ||x|-L|\cdot|\chi_J-\chi_{J_0}|dx=\int_{\mathbb{R}} ||x|-L|\cdot|\rho(x)-\chi_{J_0}|dx,$$

we get the statement of the Lemma. □

Now we turn our attention to the full energy functional for 2D Euler in S. Let

$$F(\chi_E)=\int_S\int_S \chi_E(z)\chi_E(\xi)\log(\cosh(x_1-x_2)-\cos(y_1-y_2))dzd\xi, \quad z=(x_1,y_1),\ \xi=(x_2,y_2).$$

Observe that

$$F(\chi_E)=(2\pi)^2\int_{\mathbb{R}}\int_{\mathbb{R}} \rho_E(x_1)\rho_E(x_2)|x_1-x_2|dx_1dx_2+F_1(\chi_E)-\log(2)\|\chi_E\|^2_{L^1(S)}\,, \tag{19}$$

where

$$\rho_E(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_E(x,y)dy$$

and

$$F_1(\chi_E)=\int_S\int_S \chi_E(z)\chi_E(\xi)\log(\cosh(x_1-x_2)-\cos(y_1-y_2))dzd\xi-(2\pi)^2\Phi(\rho_E)+\log(2)\|\chi_E\|^2_{L^1(S)}\,.$$

Here Φ is defined in (14).

Assume that $E\subset S$, $|E|=4\pi L$, and $|E\cap\{x>0\}|=2\pi L$, i.e., E is centered around 0.

Theorem 3.1. *There is $L_0>2$ such that for every $L>L_0$ we have*

$$|F_1(\chi_E)-F_1(\chi_{E_0})|\lesssim \int_S ||x|-L|\chi_{E\,\Delta\,E_0}dxdy\,,$$

where $E_0=[-L,L]\times\mathbb{T}$.

Proof. Consider $f\in L^1(S)\cap L^\infty(S)$ and let $f_0=\chi_{E_0}$. If $f=f_0+h$, we have

$$F_1(f)=\int_S\int_S K(z,\xi)(f_0(z)+h(z))(f_0(\xi)+h(\xi))dzd\xi$$

and

$$K(z,\xi)=\log(\cosh(x_1-x_2)-\cos(y_1-y_2))-|x_1-x_2|+\log 2. \tag{20}$$

Notice that K is symmetric and

$$\int_S K(z,\xi)f_0(\xi)d\xi=0. \tag{21}$$

Indeed,

$$\int_{-\pi}^{\pi}\log(\cosh(x_1-x_2)-\cos(y_1-y_2))dy_2=\int_{-\pi}^{\pi}\log\left(\frac{\kappa^2+1}{2\kappa}-\cos y\right)dy\,,$$

where $\kappa\ge 1$ solves equation

$$\frac{\kappa^2+1}{2\kappa}=\cosh(x_1-x_2).$$

Clearly, $\kappa=e^{|x_1-x_2|}$. We continue as

$$\int_{-\pi}^{\pi}\log\left(\frac{\kappa^2+1}{2\kappa}-\cos y\right)dy=2\int_{-\pi}^{\pi}\log|\kappa-e^{iy}|dy-\int_{-\pi}^{\pi}\log(2\kappa)dy.$$

Function $\log|\kappa-z|$ is harmonic in z in the unit disc and the mean-value theorem for harmonic functions gives

$$2\int_{-\pi}^{\pi}\log|\kappa-e^{iy}|dy-\int_{-\pi}^{\pi}\log(2\kappa)dy=2\pi\log\kappa-2\pi\log 2=2\pi|x_1-x_2|-2\pi\log 2\,.$$

Now, (21) easily follows.

Then,

$$F_1(\chi_E) - F_1(\chi_{E_0}) = \int_S \int_S K(z,\xi)h(z)h(\xi)dzd\xi$$

with $h = \chi_E - \chi_{E_0}$. Notice that $|h| = |\chi_E - \chi_{E_0}| = \chi_{E\,\Delta\,E_0}$ and let $A = E\,\Delta\,E_0$. So, it is sufficient to control $\int_A \int_A |K(z,\xi)|dzd\xi$ to complete the proof. We turn now to the kernel, $K(z,\xi)$. The following is immediate: $K(z,\xi)$ is translation invariant, i.e., $K(z,\xi) = K(z-\xi, 0)$, and also

$$|K(z,0)| \lesssim \begin{cases} e^{-0.1|z|}, & |z| > 1 \\ 1 + |\log|z||, & |z| < 1 \end{cases} \tag{22}$$

for every $z \in S$. We have the following trivial bound

$$\int_A \int_A |K(z,\xi)|dzd\xi \lesssim |A| \cdot \min(1, |A| \cdot (1 + |\log|A||)), \tag{23}$$

where we took into account two estimates:

$$\int_S |K(z,0)|dz < C \tag{24}$$

and

$$\left| \int_A \min\{1, 1 + |\log|\xi||\}d\xi \right| \lesssim |A| \cdot |\log|A||, \text{ provided that } |A| < 0.5. \tag{25}$$

The last bound follows from the observation that the maximizer for that integral is the ball centered at the origin.

Given A, we have two cases:

1. $|A| > 1$. Then

$$\int_A \int_A |K(z,\xi)|dzd\xi \lesssim |A| \lesssim \int_A ||x| - L|dxdy\,,$$

where the last inequality holds for all $A \subset S$ satisfying $|A| \geqslant 1$ (see, e.g., Lemma C.1 from Appendix C).

2. $|A| \leqslant 1$. Then, we write $A = A_1 \cup A_2$ where $A_1 = A \cap \{||x| - L| < 1\}$.

Consider A_1. If $A_1 = A_1^+ \cup A_1^-$, $A_1^+ = A_1 \cap \{|x - L| < 1\}$, $A_1^- = A_1 \cap \{|x + L| < 1\}$, then

$$\int_{A_1} \int_{A_1} |K(z,\xi)|dzd\xi \lesssim \int_{A_1} ||x| - L|dxdy$$

by Lemma C.2 and Lemma C.3 that are proved in Appendix C. The remaining terms are then bounded using (24) as follows

$$\int_{A_2} \int_{A_2} |K(z,\xi)|dzd\xi \lesssim |A_2| \lesssim \int_{A_2} ||x| - L|dxdy$$

and similarly

$$\int_{A_1} \int_{A_2} |K(z,\xi)|dzd\xi \lesssim |A_2| \lesssim \int_{A_2} ||x| - L|dxdy.$$

The proof of Theorem 3.1 is now completed. □

We define $\mathcal{M}$ as the collection of measurable sets $E \subset S$ such that E is centered around the origin, $|E| = 4\pi L$. Theorem 3.1 along with (18) and (19) give

Corollary 3.1. *There is $L_0 > 2$ such that for every $L > L_0$ and $E \in \mathcal{M}$, we have*

$$F(\chi_E) \geqslant F(\chi_{E_0}) + CL \int_{E\,\Delta\,E_0} ||x| - L|dxdy.$$

Therefore, $\arg\min_{E \in \mathcal{M}} F(\chi_E) = E_0 = [-L, L] \times \mathbb{T}$.

Now we are ready to apply our estimates to the Euler dynamics. We will start by proving the following Theorem which is identical to Theorem 1.1 except that the estimate (8) for points of centering is missing.

Theorem 3.2. *There is an absolute constant $L_0 > 2$ such that the following statement is true. If*
(a) $L > L_0$, $\epsilon < 1$,
(b) *E is a compact subset of S and 0 is one of its points of centering,*
(c) $|E| = 4L\pi$,
(d) *the regularized energy satisfies*

$$F(\chi_E) = F(\chi_{E_0}) + O(L\epsilon^2), \tag{26}$$

(e) *function θ solves 2D Euler equation* (1) *with the Biot-Savart law given by* (2) *and* (3), *then $\theta(t) = \chi_{E(t)}$ and $E(t)$ satisfies*

$$\int_S ||x - x_c(t)| - L|\chi_{E(t)\,\Delta\, E_0(t)} dxdy \lesssim \epsilon^2 \tag{27}$$

for all $t > 0$. Above $x_c(t)$ is any point of centering for $E(t)$ and $E_0(t) = [x_c(t) - L, x_c(t) + L] \times \mathbb{T}$.

Moreover, if $\mu > \epsilon$, then

$$|(E(t)\,\Delta\, E_0(t)) \cap \{||x - x_c(t)| - L| > \mu\}| \lesssim \epsilon^2\mu^{-1}.$$

Proof. Notice that $F(\chi_{E(t)})$ and $|E(t)|$ are invariants. Therefore, Corollary 3.1 gives

$$L\int_S ||x - x_c(t)| - L|\chi_{E(t)\,\Delta\, E_0(t)} dxdy \lesssim L\epsilon^2$$

and the statements follow. □

The following Lemma gives a simple geometric condition for (26) to hold.

Lemma 3.4. *If E is centered around the origin, $|E| = 4L\pi$, and $\{|x| < L-\epsilon\} \subseteq E \subseteq \{|x| < L+\epsilon\}$, then*

$$F(\chi_E) = F(\chi_{E_0}) + O(L\epsilon^2). \tag{28}$$

Proof. Consider the representation (19) for E and compare it to the same representation for E_0. For the second term, we use Theorem 3.1 to get

$$|F_1(\chi_E) - F_1(\chi_{E_0})| \lesssim \int_S ||x| - L|\chi_{E\,\Delta\, E_0} dxdy \lesssim \epsilon^2.$$

If we write $\rho_E = \chi_{J_0} + \delta$, then the first term in (19) gives

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \rho_E(x_1)\rho_E(x_2)|x_1 - x_2|dx_1dx_2 = \int_{\mathbb{R}}\int_{\mathbb{R}} (\chi_{J_0}(x_1) + \delta(x_1))(\chi_{J_0}(x_2) + \delta(x_2))|x_1 - x_2|dx_1dx_2$$

with $\|\delta\|_{L^\infty} \lesssim 1$, $\int_{\mathbb{R}^+} \delta dx = \int_{\mathbb{R}^-} \delta dx = 0$, and $\operatorname{supp}\delta \subseteq \{L - \epsilon < |x| < L + \epsilon\}$. Notice that (see, e.g., (16), (17))

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \delta(x_1)\delta(x_2)|x_1 - x_2|dx_1dx_2 \le 0.$$

For the cross product,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \delta(x_1)\chi_{J_0}(x_2)|x_1-x_2|dx_1dx_2 = \int_{-L-\epsilon}^{-L+\epsilon} \delta(x_1)\left(\int_{-L}^{L} |x_1 - x_2|dx_2\right)dx_1+\int_{L-\epsilon}^{L+\epsilon} \delta(x_1)\left(\int_{-L}^{L} |x_1 - x_2|dx_2\right)dx_1$$

Consider, e.g., the first integral. We have

$$||x_1 - x_2| - |(-L) - x_2|| \leqslant |x_1 + L| \leqslant \epsilon$$

and, therefore,

$$\int_{-L-\epsilon}^{-L+\epsilon} \delta(x_1) \left(\int_{-L}^{L} |x_1 - x_2| dx_2 \right) dx_1 = O(L\epsilon^2) + \int_{-L-\epsilon}^{-L+\epsilon} \delta(x_1) \left(\int_{-L}^{L} |(-L) - x_2| dx_2 \right) dx_1 \,.$$

Similarly,

$$\int_{L-\epsilon}^{L+\epsilon} \delta(x_1) \left(\int_{-L}^{L} |x_1 - x_2| dx_2 \right) dx_1 = O(L\epsilon^2) + \int_{L-\epsilon}^{L+\epsilon} \delta(x_1) \left(\int_{-L}^{L} |L - x_2| dx_2 \right) dx_1 \,.$$

However,

$$\int_{-L-\epsilon}^{-L+\epsilon} \delta(x_1) \left(\int_{-L}^{L} |(-L) - x_2| dx_2 \right) dx_1 = \int_{L-\epsilon}^{L+\epsilon} \delta(x_1) \left(\int_{-L}^{L} |L - x_2| dx_2 \right) dx_1 = 0 \,,$$

because $\int_{\mathbb{R}^\pm} \delta dx = 0$ and we have the statement of the Lemma. ☐

To complete the proof of Theorem 1.1, we are left with studying the dynamics of $x_c(t)$, the point of centering.

Lemma 3.5. *In the previous Theorem 3.2, a centering point $x_c(t)$ satisfies*

$$|x_c(t)| \lesssim L^{-1}\epsilon^2$$

for all times.

Proof. For the patch $\chi_E(t)$ we have also that the x-coordinate of the center of mass is conserved and equal to zero, so:

$$\int_S (x - x_c(t))\chi_{E(t)}(z) dxdy = -4\pi L x_c(t).$$

It suffices to bound the left hand side by ϵ^2. Recall $E_0(t) = [x_c(t) - L, x_c(t) + L] \times \mathbb{T}$ observe that

$$\int_S (x - x_c(t))\chi_{E(t)}(z) dxdy = \int_S (x - x_c(t))(\chi_{E(t)}(z) - \chi_{E_0(t)}(z)) dxdy.$$

We use the fact that $x_c(t)$ is the centering point for both $E(t)$ and $E_0(t)$ to write

$$\int_{x > x_c(t)} (x - x_c(t))(\chi_{E(t)}(z) - \chi_{E_0(t)}(z)) dxdy = \int_{x > x_c(t)} (x - x_c(t) - L)(\chi_{E(t)}(z) - \chi_{E_0(t)}(z)) dxdy \leqslant$$

$$\int_{x > x_c} |x - x_c(t) - L| \cdot |\chi_{E(t)}(z) - \chi_{E_0(t)}(z)| dxdy = \int_{x > x_c} |x - x_c(t) - L| \cdot \chi_{E(t)\Delta E_0(t)}(z) dxdy \lesssim \epsilon^2$$

as follows from (27). The integral over $x < x_c(t)$ is handled similarly. Thus, $|x_c(t)| \lesssim L^{-1}\epsilon^2$. ☐

The Theorem 3.2 and Lemma 3.5 give the proof of Theorem 1.1.

Appendix A. Existence and Uniqueness of Solution on S

Now we will discuss the existence and uniqueness result stated in Section 2.

Theorem A.1. *Let $\theta_0(z)$ for $z \in S = \mathbb{R} \times \mathbb{T}$ be in $L^\infty(S)$ with compact support in S. Then there exists unique (u, θ) with $u \in L^\infty(S)$ and $\theta \in L^\infty(S)$ with compact support in S such that $\partial_t \theta + u \cdot \nabla \theta = 0$ in the sense of distributions with*

$$u(z, t) = \nabla^\perp (\Gamma * \theta) = \int_S \frac{(-\sin(y - \xi_2), \sinh(x - \xi_1))}{2(\cosh(x - \xi_1) - \cos(y - \xi_2))} \theta(\xi, t) d\xi$$

and $\theta(z, 0) = \theta_0(z)$.

This Theorem is a corollary of the following result of Kelliher [6]. Note that in that work, the space $\mathcal{S}(\mathbb{R}^2)$ is the space of all divergence-free vector fields u with vorticity $\theta(u)$ so that

$$\|u\|_{L^\infty} + \|\theta(u)\|_{L^\infty} < \infty.$$

The goal is to consider bounded velocity and vorticity without assumptions on their smoothness, so $\nabla \cdot u = 0$ and $\theta(u) = \nabla \times u$ in the sense of distributions. Moreover, we say $u \in \mathcal{S}$ with vorticity θ is a bounded solution if

(1) $\partial_t \theta + u \cdot \nabla \theta = 0$ in the sense of distributions,
(2) the vorticity is transported by the flow.

Now we can state the Theorem from [6]:

Theorem A.2 (Theorem 2.9, [6]). *Assume that u^0 is in $\mathcal{S}(\mathbb{R}^2)$, let $T > 0$ be arbitrary, and fix $U_\infty(t) \in (C[0,T])^2$ with $U_\infty(0) = 0$. Let $\mathcal{K}(y) = \dfrac{y^\perp}{|y|^2}$. There exists a bounded solution u to the Euler equations in $\mathbb{R}^2$, and this solution satisfies a renormalized Biot-Savart law*

$$u(t) - u^0 = U_\infty(t) + \lim_{R\to\infty} (a_R \mathcal{K}) * (\theta(t) - \theta^0)$$

on $[0,T] \times \mathbb{R}^2$ for all smooth, compactly supported, radial cutoff functions $a_R(x) = a(x/R)$ with $a(x) = 1$ for $|x| < 1$ and $a(x) = 0$ for $|x| > 2$. This solution is unique among all solutions u with $u(0) = u^0$ that satisfy the given renormalized Biot-Savart law.

The proof of this Theorem is given in its entirety in [6]. The vector field $U_\infty(t)$ allows the work to characterize the non-uniqueness of solutions when u is only bounded. We will use this result to prove Theorem 2.1, and the choice of $U_\infty(t)$ is naturally proscribed by the boundary conditions on the stream function in (3).

Proof of Theorem 2.1. Let $\theta^0(x,y)$ on $\mathbb{R}^2$ be the periodic extension of $\theta_0(z)$ and define $u_0(z) = \nabla^\perp \Gamma * \theta_0(z)$ with periodic extension $u^0(x,y)$. Then by Theorem A.2, there exist unique u and θ defined on all of $\mathbb{R}^2$ so that

$$u(t) - u^0 = U_\infty(t) + \lim_{R\to\infty} (a_R \mathcal{K}) * (\theta(t) - \theta^0)$$

on $[0,T] \times \mathbb{R}^2$. The solution $u(x,y,t)$ and $\theta(x,y,t)$ are periodic by uniqueness.

First we will show that renormalized Biot-Savart law is equivalent to the cylindrical Biot Savart law given by (2) and (3). If we consider any $g(\xi) \in L^\infty(\mathbb{R}^2)$ such that $g(\xi) = g(\xi + (0, 2\pi))$ and $g(\xi) = 0$ for $|\xi_1| > R$, then

$$\int_{\mathbb{R}^2} a_R(z-\xi) \frac{(z-\xi)^\perp}{|z-\xi|^2} g(\xi) d\xi = \int_{\mathbb{R}} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} a_R(z-\xi_k) \frac{(z-\xi_k)^\perp}{|z-\xi_k|^2} g(\xi) d\xi\,,$$

where $\xi_k = \xi + (0, 2\pi k)$ by the periodicity of g. Observe that

$$\sum_{k=-\infty}^{\infty} \frac{(z-\xi_k)^\perp}{|z-\xi_k|^2} = \sum_{k=-\infty}^{\infty} \frac{(-(x_2 - y_2 - 2\pi k), x_1 - y_1)}{(x_1-y_1)^2 + (x_2 - y_2 - 2\pi k)^2}$$

and by Poisson's Summation Formula, we have the following identity:

Lemma A.1. *For $a, b \neq 0$*

$$\sum_{k=-\infty}^{\infty} \frac{(-(b-2\pi k), a)}{a^2 + (b-2\pi k)^2} = \frac{(-\sin(b), \sinh(a))}{2(\cosh(a) - \cos(b))}.$$

Since a_R tends to 1 uniformly and θ is compactly supported in S, we can conclude that limit in R converges to the desired cylindrical Biot-Savart law.

The boundary conditions on Ψ in (3) require that

$$\lim_{x\to+\infty} u_2(x,y) = -\lim_{x\to-\infty} u_2(x,y).$$

Since $k*\theta(x,y,t)$ satisfies this equality, it must be true that $U_\infty^{(2)}(t) = -U_\infty^{(2)}(t) = 0$. As for $U_\infty^{(1)}(t)$, observe that if $\theta_1(x,y,t)$ solves the cylindrical problem with horizontal drift

$$\partial_t\theta_1 + U_\infty^{(1)}(t)\partial_1\theta_1 + (k*\theta_1)\nabla\theta_1 = 0,$$

the translated function $\widetilde{\theta} = \theta_1(x+F(t),y,t)$ where $F(t) = \int_0^t U_\infty^{(1)}(s)ds$ satisfies

$$\partial_t\widetilde{\theta} + (k*\widetilde{\theta})\nabla\widetilde{\theta} = 0.$$

By setting $U_\infty^{(1)}(t) \equiv 0$, we factor out the possibility of a moving reference frame in the horizontal direction.

It only remains to show that the Lemma A.1 holds.

Proof of Lemma A.1. We can compute this sum explicitly. For the first component, Poisson summation formula gives

$$\sum_{k\in\mathbb{Z}} \frac{-(b-2\pi k)}{a^2+(b-2\pi k)^2} = -\frac{1}{2i}\sum_{n=1}^{\infty}(e^{ibn}e^{-|a|n} - e^{-ibn}e^{-|a|n}),$$

since we have by residue calculus

$$\int_{-\infty}^{\infty} \frac{e^{2\pi ixn}2\pi x}{a^2+(2\pi x)^2}dx = \begin{cases} \dfrac{ie^{-|a|n}}{2}, & \text{for } n>0,\\[2ex] -\dfrac{ie^{|a|n}}{2}, & \text{for } n<0,\\[2ex] 0, & n=0. \end{cases}$$

Since $|e^{-|a|\pm ib}| \leqslant e^{-|a|} < 1$ for $|a|>0$, we have

$$\sum_{k\in\mathbb{Z}} \frac{-(b-2\pi k)}{a^2+(b-2\pi k)^2} = -\frac{\sin(b)}{2(\cosh(a)-\cos(b))}.$$

For the second component, assume that $a>0$. Then,

$$\sum_{k\in\mathbb{Z}} \frac{a}{a^2+(b-2\pi k)^2} = \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} e^{-(a+ib)n} + e^{-(a-ib)n},$$

since we have

$$\int_{-\infty}^{\infty} \frac{e^{2\pi ixn}}{a^2+(2\pi x)^2}dx = \frac{e^{-|an|}}{2|a|}.$$

As before, we can use the geometric series and see that

$$\sum_{k\in\mathbb{Z}} \frac{a}{a^2+(b-2\pi k)^2} = \frac{1}{2} + \frac{1}{2}\left(\frac{e^{-(a+ib)}}{1-e^{-(a+ib)}} + \frac{e^{-(a-ib)}}{1-e^{-(a-ib)}}\right) = \frac{\sinh(a)}{2(\cosh(a)-\cos(b))}.$$

Since this expression is odd in a, it holds for $a<0$ as well. ☐

Finally, the fact that θ is periodic in the whole plane and is transported by the flow gives

$$\|\theta(t)\|_{L^1(S)} = \|\theta_0\|_{L^1(S)} \text{ and } \|\theta(t)\|_{L^\infty(S)} = \|\theta_0\|_{L^\infty(S)}.$$

By the cylindrical Biot-Savart law, we know that $\|u\|_{L^\infty(S)} \lesssim \|\theta(0)\|_{L^1(S)} + \|\theta(0)\|_{L^\infty(S)}$ and θ will remain compactly supported. $\square$

Appendix B. Existence of Minimizer

In this Appendix, we prove a standard result about existence of a minimizer in the variational problem (15).

Lemma B.1. *The problem* (15) *has a minimizer.*

Proof. Denote

$$\sigma = \inf_{\rho \in \mathcal{O}} \Phi(\rho)$$

and ρ_n is a minimizing sequence: $\Phi(\rho_n) \to \sigma$. Recall that

$$\sup_n \int (1+|x|)\rho_n dx < C. \tag{29}$$

Consider $\{\rho_n\}$. We can choose a subsequence $\{\rho_{k_n}\} \to \rho^*$ weakly over all compact sets in $\mathbb{R}$ and clearly $\rho^* \in \mathcal{O}$. Let us rename this $\{\rho_{k_n}\}$ back as $\{\rho_n\}$ for convenience. We have

$$\int_{-T}^{T} (1+|x|)\rho^* dx = \lim_{n\to\infty} \int_{-T}^{T} (1+|x|)\rho_n dx \leqslant \liminf_{n\to\infty} \int (1+|x|)\rho_n dx < C\,,$$

so

$$(1+|x|)\rho^* \in L^1(\mathbb{R}). \tag{30}$$

because T is arbitrary. Similarly, we conclude that

$$\int_0^\infty x\rho^* dx \leqslant \liminf_{n\to\infty} \int_0^\infty x\rho_n dx, \quad \int_{-\infty}^0 |x|\rho^* dx \leqslant \liminf_{n\to\infty} \int_{-\infty}^0 |x|\rho_n dx. \tag{31}$$

Notice also that

$$\int_{\mathbb{R}^+} \rho^* dx = \int_{\mathbb{R}^-} \rho^* dx = 1$$

as follows from the definition $\mathcal{O}$. We will need the following result

$$\lim_{n\to\infty} \int_0^\infty x_1\rho_n(x_1) \int_{x_1}^\infty \rho_n(x_2)dx_2dx_1 = \int_0^\infty x_1\rho^*(x_1) \int_{x_1}^\infty \rho^*(x_2)dx_2dx_1. \tag{32}$$

It is due to the tightness estimate (29), (30), weak convergence, and Dominated Convergence Theorem.

We now prove that ρ^* is a minimizer, i.e., that $\Phi(\rho^*) = \sigma$. Write $\Phi(\rho_n)$ as

$$\Phi(\rho_n) = I_1 + I_2 = \int_0^\infty \rho_n(x_1)dx_1 \int_{-\infty}^\infty |x_2 - x_1|\rho_n(x_2)dx_2 + \int_{-\infty}^0 \rho_n(x_1)dx_1 \int_{-\infty}^\infty |x_2 - x_1|\rho_n(x_2)dx_2.$$

For I_1, we have

$$I_1 = \int_0^\infty \rho_n(x_1)dx_1 \int_0^\infty |x_2 - x_1|\rho_n(x_2)dx_2 + \int_0^\infty \rho_n(x_1)dx_1 \int_0^\infty (x_2 + x_1)\rho_n(-x_2)dx_2.$$

Consider the first integral. By symmetry, it is equal to

$$2\int_0^\infty \rho_n(x_1)dx_1 \int_0^{x_1} (x_1 - x_2)\rho_n(x_2)dx_2 = 2\left(\int_0^\infty x_1\rho_n(x_1)dx_1\right)\left(\int_0^\infty \rho_n(x_2)dx_2\right) -$$
$$2\int_0^\infty \rho_n(x_1)dx_1 \int_0^{x_1} x_2\rho_n(x_2)dx_2 - 2\int_0^\infty x_1\rho_n(x_1)dx_1 \int_{x_1}^\infty \rho_n(x_2)dx_2.$$

Notice that the last two terms are equal to each other and

$$\int_0^\infty \rho^* dx = \int_0^\infty \rho_n dx = 1.$$

Thus, we are left with

$$2\left(\int_0^\infty x_1\rho_n(x_1)dx_1\right)\left(\int_0^\infty \rho^*(x_2)dx_2\right) - 4\int_0^\infty x_1\rho_n(x_1)dx_1\int_{x_1}^\infty \rho_n(x_2)dx_2 =$$

$$2\left(\int_0^\infty x_1\rho_n(x_1)dx_1\right)\left(\int_0^\infty \rho^*(x_2)dx_2\right) - 4\int_0^\infty x_1\rho^*(x_1)dx_1\int_{x_1}^\infty \rho^*(x_2)dx_2 + o(1)$$

by (32). Then, (31) applied to the first term in the last expression gives

$$\int_0^\infty \rho^*(x_1)dx_1\int_0^{x_1}(x_1-x_2)\rho^*(x_2)dx_2 \leqslant \liminf_{n\to\infty}\int_0^\infty \rho_n(x_1)dx_1\int_0^{x_1}(x_1-x_2)\rho_n(x_2)dx_2.$$

In a similar way, one shows that

$$\int_0^\infty\int_0^\infty \rho^*(x_1)\rho^*(-x_2)(x_1+x_2)dx_1dx_2 \leqslant \liminf_{n\to\infty}\int_0^\infty\int_0^\infty \rho_n(x_1)\rho_n(-x_2)(x_1+x_2)dx_1dx_2.$$

The integral I_2 can be handled similarly. Adding up these inequalities, we see that $\Phi(\rho^*) \leqslant \sigma$ so ρ^* is a minimizer.

□

Appendix C. Three auxiliary Lemmas

In this Appendix, we will prove three results used in the main text. We introduce notation $\square = [-1,1]\times\mathbb{T}$.

Lemma C.1. *We have*

$$\min_{Q\subset S, |Q|=s}\int_\square |x|\chi_Q dxdy = \frac{s^2}{8\pi}$$

and the minimum is achieved on $Q_{min} = [-s/4\pi, s/4\pi]\times\mathbb{T}$.

Proof. The result follows immediately from the structure of the weight $|x|$ against which χ_Q is being integrated. □

Recall the kernel $K(z,\xi)$ which was introduced in (20).

Lemma C.2. *Let* $L>2$. *If* $A_1^+\subset\{|x-L|<1\}$ *and* $A_1^-\subset\{|x+L|<1\}$, *then*

$$\int_{A_1^+}\int_{A_1^-}|K(z,\xi)|dzd\xi \lesssim \int_{A_1^+\cup A_1^-}||x|-L|dxdy. \tag{33}$$

Proof. The estimate (22) implies that

$$\int_{A_1^+}\int_{A_1^-}|K(z,\xi)|dzd\xi \lesssim |A_1^+|\cdot|A_1^-| \leqslant 0.5(|A_1^+|^2+|A_1^-|^2).$$

Now, to estimate $|A_1^\pm|^2$, we only need to change variables as $x\pm L=\widehat{x}$ in the right-hand side of (33) and notice that

$$|B|^2 \lesssim \int_S |\widehat{x}|\chi_B d\widehat{z}$$

for every measurable set $B\subset\{|\widehat{x}|<1\}$ by Lemma C.1. □

Lemma C.3. *If A is a measurable subset of $\square$, then*

$$\int_A\int_A |\log|z-\xi||dzd\xi \lesssim \int_\square |x|\chi_A dxdy. \tag{34}$$

Proof. Notice that

$$\int_A\int_A |\log|z-\xi||dzd\xi \leqslant \int_\square\int_\square |\log|z-\xi||dzd\xi < \infty\,.$$

Therefore, by Lemma C.1 applied with $s=|A|$, we can always assume that $|A|=\epsilon<\epsilon_0$ where ϵ_0 is sufficiently small. Consider $E_j=A\cap\{j\epsilon<x<(j+1)\epsilon\}$ and let $\delta_j=|E_j|$, $I_j=\delta_j/\epsilon, j=-N,\ldots,N, N=[\epsilon^{-1}]$. We have $\sum_{j=-N}^N I_j=1$.

In (34),

$$\int_\square |x|\chi_A dxdy \sim \epsilon^2\sum_{j=-N}^N |j|I_j+\epsilon^2. \tag{35}$$

Indeed, if $j\neq 0,-1$, then

$$\int_{E_j} |x|\chi_A dxdy \sim \epsilon|j|\delta_j=\epsilon^2|j|I_j.$$

For $j=0$ and $j=-1$, we have

$$\int_{E_j} |x|\chi_A dxdy \leqslant \epsilon\delta_j\leqslant\epsilon^2$$

and

$$\int_\square |x|\chi_A dxdy \lesssim \epsilon^2\sum_{j=-N}^N |j|I_j+\epsilon^2$$

follows. To prove a lower bound, notice that $\delta_0+\delta_{-1}>\epsilon/2$ implies

$$\int_{E_0\cup E_{-1}} |x|\chi_A dxdy \gtrsim \epsilon^2$$

by applying Lemma C.1 with $s=\delta_0+\delta_{-1}$. If $\delta_0+\delta_{-1}<\epsilon/2$, then $|A\backslash(E_0\cup E_{-1})|>\epsilon/2$ and

$$\int_{|x|>\epsilon} |x|\chi_A dxdy \gtrsim \epsilon^2.$$

Therefore, we have

$$\int_\square |x|\chi_A dxdy \gtrsim \epsilon^2 \tag{36}$$

either way. Moreover, from the definition of δ_j and I_j, we get

$$\int_\square |x|\chi_A dxdy \geqslant \sum_{j\neq\{-1,0\}}\int_\square |x|\chi_{E_j}dx \gtrsim \epsilon^2\sum_{j\neq\{-1,0\}}|j|I_j\,. \tag{37}$$

Taking the sum of (36) and (37), we get

$$\int_\square |x|\chi_A dxdy \gtrsim \sum_{j\neq\{-1,0\}}|j|I_j+\epsilon^2 \gtrsim \epsilon^2\sum_{j=-N}^N |j|I_j+\epsilon^2 \tag{38}$$

since $\epsilon^2\geqslant\epsilon^2 I_{0(-1)}$.

Define the potential $U(z)=\int_A|\log|z-\xi||d\xi$. For fixed $\{\delta_l\}, l=-N,\ldots,N$, we will estimate $\max_{E_j}U(z)$ for each $j=-N,\ldots,N$. Let

$$\max_{E_j}U(z)=U(z_j^*).$$

We can bound the right hand side above by decomposing $U(z_j^*)$ as

$$U(z_j^*) \leqslant M_1 + M_2 + M_3.$$

The term M_1 comes from considering the ϵ-ball around the point of maximum z_j^*. It satisfies

$$M_1 \lesssim \int_{|z|<\epsilon} |\log|z||dz \sim \epsilon^2 |\log \epsilon|. \tag{39}$$

The term M_2 comes from integrating over $(E_{j-1} \cup E_j \cup E_{j+1}) \backslash B_\epsilon(z_j^*)$. To estimate it, we notice the following. Consider, e.g, E_{j-1}. If $\delta_{j-1} = |E_{j-1}| < \epsilon^2$, then

$$\int_{E_{j-1}} |\log|\xi - z_j^*||d\xi \lesssim \epsilon^2 |\log \epsilon| ,$$

because the maximizer of the integral in the left-hand side belongs to a ball $|\xi - z_j^*| \lesssim \epsilon$. On the other hand, if $|E_{j-1}| > \epsilon^2$, then

$$\int_{E_{j-1}} |\log|\xi - z_j^*||d\xi \lesssim \epsilon \int_0^{I_{j-1}} |\log y|dy \lesssim \epsilon I_{j-1}(|\log I_{j-1}| + 1) ,$$

because the maximizer of the integral belongs to the rectangle of hight $\sim I_{j-1}$. Arguing similarly for E_j and E_{j+1}, we get

$$M_2 \lesssim \epsilon(\epsilon|\log \epsilon| + I_j|\log I_j| + I_{j-1}|\log I_{j-1}| + I_{j+1}|\log I_{j+1}| + I_{j-1} + I_j + I_{j+1}) \lesssim \epsilon , \tag{40}$$

where the last bound follows from $x|\log x| \lesssim 1$ when $x < 1$. Finally, the term M_3 covers integration over the remaining E_k. It will satisfy

$$M_3 \lesssim \sum_{k:|k-j|>1,|k|\leqslant N} \left(\epsilon^2 |\log(\epsilon|k-j|)| + \epsilon \int_0^{I_k} |\log(y + \epsilon|k-j|)|dy \right) .$$

Indeed,

$$|\log|z_j^* - \xi|| \lesssim |\log(|k-j|\epsilon + |y_j^* - \xi_2|)|, \quad \xi \in E_k ,$$

where $z_j^* = (x_j^*, y_j^*), \xi = (\xi_1, \xi_2)$. Then, we have an upper bound for the following variational problem

$$\sup_{\Upsilon_k \subseteq \{k\epsilon < x < (k+1)\epsilon\}, |\Upsilon_k| = |E_k|} \int_{\Upsilon_k} |\log(|k-j|\epsilon + |\xi_2|)|d\xi \lesssim \epsilon^2 |\log(\epsilon|k-j|)| + \epsilon \int_0^{I_k} |\log(y + \epsilon|k-j|)|dy , \tag{41}$$

where the first term comes from the case when $|E_k| \leqslant \epsilon^2$, the second one comes from the other case and the observation that the optimal configuration Υ_k^* is a rectangle of the size $\sim I_k$.

Integration gives

$$\int \log y dy = y \log y - y + C$$

and so

$$\int_0^{I_k} |\log(y + \epsilon|k-j|)|dy \leqslant \int_0^{I_k} |\log y|dy \lesssim I_k |\log I_k| + I_k.$$

The first term in the right-hand side of (41) gives

$$\epsilon^2 \sum_{k:|k-j|>1,|k|\leqslant N} |\log(\epsilon|k-j|)| \lesssim \epsilon^2 \sum_{1<|l|\leqslant 2N} |\log(\epsilon|l|)| \lesssim \epsilon \int_{|x|<1} |\log|x||dx \lesssim \epsilon$$

by making comparison to an integral. Finally, summing over k, we have

$$M_3 \lesssim \epsilon + \epsilon \sum_{k=-N}^{N} I_k |\log I_k|.$$

Taking into account (39),(40), we get

$$\max_{z\in A} U(z) \lesssim \epsilon + \epsilon \sum_{k=-N}^{N} I_k |\log I_k|.$$

Then,

$$\int_A \int_A |\log|z-\xi||dzd\xi \lesssim \epsilon^2 + \epsilon^2 \sum_{k=-N}^{N} I_k |\log I_k|.$$

The trivial estimate

$$|u\log u| \leqslant C(\gamma)u^\gamma, \quad 0<\gamma<1, 0<u<1$$

implies

$$\sum_{j\neq 0} I_j|\log I_j| \lesssim \sum_{j\neq 0} I_j^\gamma = \sum_{j\neq 0} (jI_j)^\gamma j^{-\gamma} \leqslant \left(\sum_{j\neq 0}|j|I_j\right)^{1/p} \left(\sum_{j\neq 0}|j|^{-\gamma p'}\right)^{1/p'} \lesssim \left(\sum_{j\neq 0}|j|I_j\right)^{\gamma}$$

by Hölder inequality with $p = 1/\gamma$ and $\gamma > 0.5$. We have

$$\epsilon^2 I_0|\log I_0| + \epsilon^2 \left(\sum_{j\neq 0}|j|I_j\right)^{\gamma} \lesssim \epsilon^2 + \epsilon^2 \sum_{j\neq 0}|j|I_j$$

and application of (35) finishes the proof of Lemma C.3. □

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