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Score

Instructions

- Write neatly on this exam. If you need extra paper, let us know.
- On Problems 1, 2, and 3, only the answer will be graded.
- On Problems 4, 5, 6, and 7 you must show your work and we will grade the work and your justification, and not just the final answer.
- Each problem worth either 14 or 15 points.
- No calculators, books, or notes (except for those notes on your 3 inch by 5 inch notecard.)
- Please simplify any formula involving a trigonometric function and an inverse trigonometric function. For example, please write \( \cos(\arcsin x) = \sqrt{1-x^2} \).
- Final answers should not involve functions applied to either infinity or applied to a point outside of their domain. For instance, \( \arctan(\infty) \) and \( \ln(0) \) and \( \sqrt{-5} \) will not be accepted as a final answer. In a question involving a limit, we will accept a final answer of “\( \infty \)” as synonymous with “The limit does not exist”.

Formulas

You may freely quote any algebraic or trigonometric identity, as well as any of the following formulas or minor variants of those formulas.

- $\int x^n \, dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & \text{when } n \neq -1 \\ \ln |x| + C & \text{when } n = -1 \end{cases}$
- $\int e^x \, dx = e^x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sin x \, dx = -\cos x + C$
- $\int \tan x \, dx = -\ln |\cos x| + C$
- $\int \cot x \, dx = \ln |\sin x| + C$
- $\int \sec x \, dx = \ln |\sec x + \tan x| + C$
- $\int \csc x \, dx = -\ln |\csc x + \cot x| + C$
- $\int \frac{1}{1+x^2} \, dx = \arctan(x) + C$. 
1. For each statement below, CIRCLE true or false.

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<th>(a) True False</th>
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(a) There exist unique numbers $A, B, C$ and $D$ so that

$$
\int \frac{1}{(x-7)^2(2x-9)} \, dx = \int A \frac{x}{x-7} + B + C (x-7) + D \frac{2x-9}{2x-9} \, dx.
$$

(b) $\int_1^\infty \frac{t^2}{t^3 + e^t} \, dt$ is a finite number.

(c) $\int_0^2 \sqrt{4-x^2} \, dx > 4$.

(d) $\int_0^\infty \frac{x+1}{x^{2/3} + x^2} \, dx$ is a finite number.

(e) If $I_n = \int \sec^n x \, dx$ then $I_2 = \tan(x) + C$.

Solution:

(a) False.
(b) True.
(c) False.
(d) True.
(e) True.
2. On this page, only the answer will be graded.

(a) Compute \( \int \frac{dx}{3+7x^2} \).

**Solution:** We want \( 3 + 7x^2 = 3 + 3z^2 \) so we substitute \( x = \sqrt{\frac{3}{7}}z \). This yields \( dx = \sqrt{\frac{3}{7}} \, dz \). This yields:

\[
\int \frac{dx}{\sqrt{3 + 7x^2}} = \int \frac{\sqrt{\frac{3}{7}} \, dz}{3 + 3z^2} = \frac{\sqrt{3}}{3\sqrt{7}} \int \frac{dz}{1 + z^2} = \frac{\sqrt{3}}{3\sqrt{7}} \arctan(z) + C = \frac{\sqrt{3}}{3\sqrt{7}} \arctan\left(\sqrt{\frac{7}{3}}x\right) + C
\]

The answer is \( \frac{\sqrt{3}}{3\sqrt{7}} \arctan\left(\sqrt{\frac{7}{3}}x\right) + C \)

(b) Compute \( \int x^{2013} \ln(x) \, dx \).

**Solution:** Use integration by parts with \( f = \ln(x) \) and \( g' = x^{2013} \). Then \( f' = \frac{1}{x} \) and \( g = \frac{x^{2014}}{2014} \) and we get:

\[
\int x^{2013} \ln(x) \, dx = \int fg' = fg - \int f'g
\]

\[
= \frac{\ln(x)x^{2014}}{2014} - \int \frac{1}{x} \cdot \frac{x^{2014}}{2014} \, dx = \frac{\ln(x)x^{2014}}{2014} - \frac{1}{2014} \int x^{2013} \, dx = \frac{\ln(x)x^{2014}}{2014} - \frac{x^{2014}}{2014^2} + C
\]

The answer is \( \frac{\ln(x)x^{2014}}{2014} - \frac{x^{2014}}{2014^2} + C \)
(a) Compute \( \int (1 + \sin(3\theta))^2 d\theta \).

Solution:

\[
\int (1 + \sin(3\theta))^2 d\theta = \int 1 + 2 \sin(3\theta) + \sin^2(3\theta) d\theta
\]
\[
= \int 1 + 2 \sin(3\theta) + \frac{1}{2} (1 - \cos(6\theta)) d\theta
\]
\[
= \int \frac{3}{2} + 2 \sin(3\theta) - \frac{1}{2} \cos(6\theta) d\theta
\]
\[
= \frac{3\theta}{2} - \frac{2 \cos(3\theta)}{3} - \frac{1}{12} \sin(6\theta) + C
\]

The answer is: \( \frac{3\theta}{2} - \frac{2 \cos(3\theta)}{3} - \frac{1}{12} \sin(6\theta) + C \) or any expression which is equivalent via trigonometric identities.

(b) Consider the improper integral \( \int_a^\infty \frac{1}{x(x+1)(3x-11)(2x-57)} dx \). Find some \( a > 0 \) such that this improper integral equals a finite number.

Solution: These vertical asymptotes lie at \( x = 0, x = -1, x = 11/3, x = 57/2 \). So if \( a > 57/2 \), then we only need to worry about the behavior at infinity. But by the limit comparison test, this integral will behave like the integral of \( \frac{1}{x^4} \) as \( x \to \infty \).

Thus any \( a > 57/2 \) work.
4. On this page partial credit is available.

(a) Compute $\int \frac{x^3 + x^2}{x^2 + 2} \, dx$.

**Solution** Notice that the degree of the numerator is greater than or equal to the degree of the denominator, so we need to do some kind of polynomial division to get this into a form we can work with. We compute

$$\frac{x^3 + x^2}{x^2 + 2} = x + 1 - \frac{2x + 2}{x^2 + 2}$$

We then have:

$$\int \frac{x^3 + x^2}{x^2 + 2} \, dx = \int x + 1 - \frac{2x + 2}{x^2 + 2} \, dx$$

$$= \frac{x^2}{2} + x - \int \frac{2}{x^2 + 2} \, dx - \int \frac{2x}{x^2 + 2} \, dx$$

Now we solve the other integrals separately.

$$\int \frac{2}{x^2 + 2} \, dx \int \frac{2}{2z^2 + 2} (\sqrt{2} \, dz) \quad \sqrt{2}z = x \quad \sqrt{2} \, dz = dx$$

$$= \sqrt{2} \int \frac{1}{z^2 + 1} \, dz$$

$$= \sqrt{2} \arctan(z) + C$$

$$= \sqrt{2} \arctan\left(\frac{1}{\sqrt{2}} x\right) + C$$

For the other integral we get:

$$\int \frac{2x}{x^2 + 2} \, dx = \int \frac{1}{u} \, du \quad u = x^2 + 2$$

$$= \ln |u| + C$$

$$= \ln(x^2 + 2) + C$$

Putting this all together, our final answer is: $\frac{x^2}{2} + x - \sqrt{2} \arctan\left(\frac{1}{\sqrt{2}} x\right) - \ln(x^2 + 2) + C$

(b) Compute $\int_{4}^{\infty} \frac{5}{(2x - 1)(x + 2)} \, dx$.

**Solution:** We rewrite this in the form:

$$\frac{5}{(2x - 1)(x + 2)} \, dx = \frac{A}{2x - 1} + \frac{B}{x + 2}.$$ 

Clearing denominators we get $5 = A(x + 2) + B(2x - 1) = (A + 2B)x + (2A - B)$. We thus have $0 = A + 2B$ and $5 = 2A - B$ which yields $A = 2$ and $B = -1$. 
\[
\int_4^\infty \frac{5}{(2x-1)(x+2)} \, dx = \int_4^\infty \frac{2}{2x-1} - \frac{1}{x+2} \, dx
\]

\[
= \lim_{b \to \infty} \int_4^b \frac{2}{2x-1} - \frac{1}{x+2} \, dx
\]

\[
= \lim_{b \to \infty} [\ln |2x-1| - \ln |x+2|]_4^b
\]

\[
= \lim_{b \to \infty} \ln \left| \frac{2x-1}{x+2} \right|
\]

\[
= \lim_{b \to \infty} \ln \left| \frac{2b-1}{b+2} \right| - \ln \frac{7}{6}
\]

\[
= \ln \left( \lim_{b \to \infty} \frac{2b-1}{b+2} \right) - \ln \frac{7}{6}
\]

\[
= \ln (2) - \ln \frac{7}{6}.
\]

The answer is \(\ln (2) - \ln \frac{7}{6}\) though there are other ways to write this, using logarithmic identities.
5. On this page partial credit is available.

Compute \( \int \frac{1}{t \ln^4(t) \sqrt{\ln^2(t) - 1}} \, dt \).

**Solution:**

\[
\int \frac{1}{t \ln^4(t) \sqrt{\ln^2(t) - 1}} \, dt = \int \frac{1}{u^4 \sqrt{u^2 - 1}} \, du \\
= \int \frac{\sec(\theta) \tan(\theta)}{\sec^4(\theta) \sqrt{\sec^2(\theta) - 1}} \, d\theta \\
= \int \frac{d\theta}{\sec^3 \theta} \\
= \int \cos^3 \theta d\theta \\
= \int \cos \theta (1 - \sin^2 \theta) d\theta \\
= \int (1 - w^2) dw \\
= w - \frac{w^3}{3} + C \\
= \sin \theta - \frac{\sin^3 \theta}{3} + C
\]

Since \( u = \sec \theta \) we can use a triangle picture to get \( \sin \theta = \frac{\sqrt{u^2 - 1}}{u} \)

\[
= \frac{\sqrt{u^2 - 1}}{u} - \frac{1}{3} \left( \frac{\sqrt{u^2 - 1}}{u} \right)^3 + C \\
= \frac{\sqrt{\ln(t)^2 - 1}}{\ln(t)} - \frac{1}{3} \left( \frac{\sqrt{\ln(t)^2 - 1}}{\ln(t)} \right)^3 + C
\]

So the answer is \( \frac{\sqrt{\ln(t)^2 - 1}}{\ln(t)} - \frac{1}{3} \left( \frac{\sqrt{\ln(t)^2 - 1}}{\ln(t)} \right)^3 + C \).
6. On this page partial credit is available.

Find positive numbers $A$ and $B$ so that $A \leq \int_1^{\infty} \frac{1 + \sin^2(x)}{x^3 + x} \, dx \leq B$. Justify your answer.

**Solution:** Lower bound: We can make the fraction smaller by making the numerator smaller and the denominator bigger. We have $1 \leq 1 + \sin^2(x)$ for all $x$ and $x^3 + x \geq 2x^3$ for all $x \geq 1$. Thus

$$\int_1^{\infty} \frac{1}{2x^3} \leq \int_1^{\infty} \frac{1 + \sin^2(x)}{x^3 + x} \, dx$$

We compute that

$$A = \int_1^{\infty} \frac{1}{2x^3} = \frac{1}{2} \lim_{b \to \infty} \int_1^{b} \frac{1}{x^3} \, dx$$

$$= \frac{1}{2} \lim_{b \to \infty} \left[-\frac{1}{2} x^{-2}\right]_1^b$$

$$= \frac{1}{2} \lim_{b \to \infty} \left[-\frac{1}{2} b^{-2} + \frac{1}{2}\right]$$

$$= \frac{1}{4}.$$ 

Upper bound: We can make the fraction bigger by making the numerator bigger and the denominator smaller. We have $1 + \sin^2(x) \leq 2$ for all $x$ and $x^3 + x \geq x^3$ for all $x \geq 0$. Thus

$$B = \int_1^{\infty} \frac{1 + \sin^2(x)}{x^3 + x} \, dx \leq \int_1^{\infty} \frac{2}{x^3} \, dx.$$ 

By the above computation, we see that $\int_1^{\infty} \frac{2}{x^3} \, dx = 4 \cdot \int_1^{\infty} \frac{1}{x^3} = 4 \cdot \frac{1}{4} = 1.

We thus have $A = \frac{1}{4}$ and $B = 1$. **Note:** there are many other possible answers here. The question is all about the validity of your justification, not the final answer.
Let \( I_n = \int \sin^n x \, dx \). Use the reduction formula
\[
I_n = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}
\]
to compute \( \int_0^{2\pi} \sin^8 x \, dx \).

**Solution:** First we compute a version of the reduction formula for the definite integrals \( A_n = \int_0^{2\pi} \sin^n x \, dx \). We obtain this by adding bounds to the given reduction formula, so that we get:
\[
A_n = \left[ -\frac{1}{n} \sin^{n-1} x \cos x \right]_0^{2\pi} + \frac{n-1}{n} A_{n-2}
= \left[ -\frac{1}{n} \sin^{n-1}(2\pi) \cos(2\pi) + \frac{1}{n} \sin^{n-1}(0) \cos(0) \right] + \frac{n-1}{n} A_{n-2}
= [0] + \frac{n-1}{n} A_{n-2}
= \frac{n-1}{n} A_{n-2}
\]
We first compute \( A_0 \) and then we will apply this new reduction formula repeatedly.
\[
A_0 = \int_0^{2\pi} \sin^0 x \, dx = \int_0^{2\pi} 1 \, dx = [x]_0^{2\pi} = 2\pi.
\]
Now we compute \( A_2, A_4 \) and \( A_6 \) based on \( A_0 \) and on our formula \( A_n = \frac{n-1}{n} A_{n-2} \).
- \( A_2 = \frac{1}{2} A_0 = \pi \).
- \( A_4 = \frac{3}{4} A_2 = \frac{3}{4} \pi \).
- \( A_6 = \frac{5}{6} A_4 = \frac{5}{4} \pi \).
- \( A_8 = \frac{7}{8} A_6 = \frac{7}{6} \pi \).

So we conclude that
\[
A_8 = \int_0^{2\pi} \sin^8 x \, dx = \frac{3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8} \pi = \frac{35}{64} \pi.
\]