Research Statement

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My work has focused on understanding the topology of spectral covers - certain finite covers that arise in the “classical limit” of the geometric Langlands program - by studying the classifying space of spectral covers. The latter is a stratified space (more exactly, an Artin stack) with some nice properties that make its rational cohomology ring computable. I intend to apply certain tricks involved in this computation to other stratified stacks.

Spectral Covers

We work always over \(\mathbb{C}\).

**Definition 1.** A spectral cover \(\bar{S} \to S\) is a finite scheme map which, étale-locally on \(S\), embeds in \(\mathbb{A}^1_S\).

**Example 1.**
- Any unramified finite cover is a spectral cover.
- \(\text{Spec } \mathbb{C}[x] \to \text{Spec } \mathbb{C}[x^2]\) is a spectral cover.
- Let \(\mathcal{L}\) be a line bundle on \(S\). We can form the “\(n\)-th infinitesimal neighborhood of the zero section of \(|\mathcal{L}^n|\)” more precisely relative spec of the sheaf of algebras \(\text{Sym}^n \mathcal{L} / \text{Sym}^n \mathcal{L}\). This is a spectral cover.

These examples show that spectral covers are topologically interesting. In particular, the last example shows that spectral covers are at least as interesting as line bundles. Note that a spectral cover need not globally embed in any line bundle.

Though spectral covers make sense from a classical perspective (as polynomials in one variable, considered up to change in coordinate), the main motivation for spectral covers comes from Higgs bundles. A Higgs bundle on \(S\) is a family of square matrices, that is, a locally free sheaf \(\mathcal{F}\) with a (twisted) endomorphism \(x : \mathcal{F} \to \mathcal{F} \otimes \mathcal{L}\) for some line bundle \(\mathcal{L}\). The spectrum of \(x\), or more exactly the moduli space of eigenvalues of \(x\), is a spectral cover embedding in \(\mathcal{L}\). Note that this can be non-reduced.

**Example 2.** Let
\[
    x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \text{End } \mathbb{C}^{\oplus 2}
\]
and let \(\mathfrak{a} \subset \mathfrak{gl}\) be the centralizer of \(x\). This defines a Higgs bundle on \(\text{Spec } \mathbb{C}\). It has an eigenspace over \(\mathbb{C}[\epsilon]/(\epsilon^2)\) generated by \([1, \epsilon]\). In fact the associated spectral cover is \(\mathbb{C}[\epsilon]/(\epsilon^2)\).

The functor from Higgs bundles to spectral covers can thought of as a version of the Hitchin map for \(GL_n\). It can be used to study moduli of Higgs bundles, since specifying a Higgs bundle with a fixed spectral cover requires a lot less data than specifying an arbitrary Higgs bundle (there is a more precise statement in the next section).

For many applications it is convenient to consider a floppier notion, where, instead of selecting a twisted endomorphism \(x : \mathcal{F} \to \mathcal{F} \otimes \mathcal{L}\), one selects a sheaf of commuting operators \(\mathfrak{a} \subset \text{End } \mathcal{F}\) which is locally
generated by a single element. This is called an abstract regular Higgs bundle. An abstract regular Higgs bundle still gives rise to a spectral cover, defined as the moduli space of eigenspaces of $a$. We will return to this in the next section.

Our goal is to define cohomological invariants of spectral covers. By analogy with the construction of the Chern classes as cohomology classes of $BGL_n$, we do this by computing the cohomology ring of the classifying space of degree-$d$ spectral covers. That is, let $\mathcal{M}$ be the stack such that $\text{Mor}(S, \mathcal{M})$ is the category of degree-$d$ spectral covers of $S$. I have shown:

**Theorem 1.** $H^*(\mathcal{M}, \mathbb{Q})$ has a basis corresponding to weighted partitions of $n$, that is, partitions $\lambda_1 + \ldots + \lambda_l = n$ where each $\lambda_i$ is given an integer weight $\mu_i$, with $\mu \geq \lambda_i - 1$ and $\mu = 0$ if $\lambda_i = 1$. The cohomological degree is twice the sum of the weights. The cup product can be expressed in terms of “merging the partitions and adding the weights.”

The big idea is to pass to a ramified $\Sigma_n$-cover $C \to \mathcal{M}$ called the “universal cameral cover” (this will be explained in the next section). One observes that $C$ has a stratification by classifying spaces, and we show that each is the classifying space of an extension of a torus by a unipotent group. In particular its cohomology ring is a polynomial ring. We then use the connecting triangle (or the Puppe sequence, depending on your point of view) to compute $H^*(C, \mathbb{Z})$. Finally we use transfer to obtain $H^*(\mathcal{M}, \mathbb{Q})$.

**Cameral Covers and Characteristic Classes of Higgs Bundles**

Our results are better stated in terms of “cameral covers”. Cameral covers, introduced by Donagi, solve two problems. First, it is often helpful when studying fiber bundles with structure group $W$, to consider $W$-torsors instead. Aside from ramification, spectral covers are fiber bundles with structure group $\Sigma_n$, so to study them one would like some notion of “ramified $\Sigma_n$-torsors”. Second, the Higgs bundles described above have something to do with $\text{ramified } \Sigma_n$-torsors. In particular its cohomology ring is a polynomial ring. We then use the connecting triangle (or the Puppe sequence, depending on your point of view) to compute $H^*(C, \mathbb{Z})$. Finally we use transfer to obtain $H^*(\mathcal{M}, \mathbb{Q})$.

**Definition 2.** Let $t$ be a vector space and $W \subset GL(t)$ be a finite group generated by reflections\(^1\). An $S$-scheme $C \to S$ with $W$-action is a $(W, t)$-cameral cover if étale-locally it is the pullback of $t \to t/W$ along some map $S \to t/W$.

Any $W$-torsor is a cameral cover, but cameral covers may be ramified if they have points with non-trivial stabilizer in $W$. Donagi has shown that the categories of degree-$n$ spectral covers and $(\Sigma_n, \mathbb{C}^{\Sigma_n})$ cameral covers are equivalent.

Let $\mathcal{M}$ be the stack of $(W, t)$-cameral covers, $\mathcal{A}$ be the hyperplane arrangement of reflecting hyperplanes for the action of $W$ on $t$. Let $L(\mathcal{A})$ be the set of intersections of hyperplanes. It is a monoid under intersection. I define a weighted version $L^\mu(\mathcal{A})$ as follows. Call $X \in L(\mathcal{A})$ irreducible if the hyperplane arrangement $\mathcal{A}_X$, consisting of hyperplanes of $\mathcal{A}$ that contain $X$, is an irreducible hyperplane arrangement.

**Definition 3.** Let $L^\mu(\mathcal{A})$ be the free abelian monoid on the tuples $(X, \mu)$ where $X \in L(\mathcal{A})$ is irreducible, $\mu \geq \text{codim } X$ is an integer, and $\mu = 0$ whenever $X = t$, modulo the relation $\prod_{i=1}^l (X_i, \mu_i) = \left( \bigcap_{i=1}^l X_i, \sum_{i=1}^l \mu_i \right)$ whenever $\bigcap_{i=1}^l X_i$ is irreducible.

Forgetting the weights $\mu$ gives a surjective monoid morphism $L^\mu(\mathcal{A}) \to L(\mathcal{A})$. Let $\mathcal{M}$ be the classifying space of $(W, t)$-cameral covers. I show:

**Theorem 2.** $H^*(\mathcal{M}, \mathbb{Q}) \cong \mathbb{Q}[L^\mu(\mathcal{A})]^W$ as graded rings.

\(^1\)In fact, our results hold verbatim if we allow $W$ to be generated by complex reflections - that is, diagonalizable matrices with at most one eigenvalue not equal to 1.
Again the idea is to compute $H^*(C, \mathbb{Z})$ (where $C \to \mathcal{M}$ is the universal cameral cover) and then use transfer. I can also show that $H^*(C, \mathbb{Z}) \cong K_0 C$, but so far this has not given useful information about $K_0 \mathcal{M}$.

Let $G$ be a connected reductive algebraic group. Suppose given a $G$-torsor $E_G \to S$. Associated to this is a sheaf of Lie algebras $\mathfrak{e}$ on $S$. Let $\mathfrak{a}$ be a sheaf of commutative subalgebras of $\mathfrak{e}$. If $\mathfrak{a}$ is locally the centralizer of a regular element of $\mathfrak{e}$ then we call the pair $(E_G, \mathfrak{a})$ an abstract regular $G$-Higgs bundle. These were introduced by Donagi and Gaitsgory. Given an abstract regular Higgs bundle $(E_G, \mathfrak{a})$ one obtains a $(W, t)$ cameral cover $C \to S$, where $W$ is the Weyl group of $G$ and $t$ is the Lie algebra of a maximal torus, by letting $C$ be the moduli space of Borel subalgebras of $\mathfrak{e}$ containing $\mathfrak{a}$. This generalizes the functor from Higgs bundles to spectral covers. Donagi and Gaitsgory have shown that the category of abstract regular $G$-Higgs bundles with cameral cover $C \to S$ is a gerbe for a sheaf of abelian groups on $C$, and have described this gerbe explicitly. It is in this sense that specifying a Higgs bundle with fixed cameral cover is easier than specifying an arbitrary Higgs bundle.

Donagi-Gaitsgory’s description is powerful but very complicated. I use their description to get much coarser information - the rational cohomology of the stack of Higgs bundles specifying an arbitrary Higgs bundle.

I can also show that

$$H^*(\mathcal{H}, \mathbb{Q}) = (\mathbb{Q}[L^\mu(A)][x_1, \ldots, x_l]/I)^W$$

as graded rings.

**Future plans**

**Cohomology of \{finite covers that locally embed in $\mathbb{A}^2$ \}**

A natural generalization of spectral covers is finite covers $\tilde{S} \to S$ that étale-locally embed in $\mathbb{A}^2_S$ rather than $\mathbb{A}^2_S$. Call such a cover “planar” and let $\mathcal{N}$ be the classifying space of degree-$n$ planar covers. Write $\text{Hilb}^n \mathbb{A}^2$ for the Hilbert scheme of $n$ points in the plane, parametrizing finite subschemes of $\mathbb{A}^2$. Forgetting the embedding in $\mathbb{A}^2$ defines a smooth, schematic map to $\mathcal{N}$, and since $\text{Hilb}^n \mathbb{A}^2$ is smooth, this implies that $\mathcal{N}$ is smooth as well. Note that the space of spectral covers $\mathcal{M}$ is dense in $\mathcal{N}$.

The ring $H^*(\text{Hilb}^n, \mathbb{Q})$ is very interesting, so $H^*(\mathcal{N}, \mathbb{Q})$ ought to be interesting too. The main things that made computation of $H^*(\mathcal{M}, \mathbb{Q})$ possible were

1. $\mathcal{M}$ stratifies into finitely many classifying spaces, indexed by partitions.

2. Automorphism groups of points of $\mathcal{M}$ are of the form “unipotent group extending a torus extending finite group”.

3. There exists a flat cover $C \to \mathcal{M}$, such that the pullbacks of the strata to $C$ behave like unions of linear spaces.

$\mathcal{N}$ has similar properties, but not quite as good. Analogously to (1), $\mathcal{N}$ has a stratification indexed by partitions, though the strata are no longer classifying spaces. Analogously to (2), automorphisms groups of points of $\mathcal{N}$ are of the form “unipotent group extending $H$ extending a finite group” where $H$ is a product of connected reductive subgroups of $GL_2$.

Ingredient (3) is less encouraging. $C \to \mathcal{M}$ is obtained by descending the flat map $\mathbb{A}^n \to \mathbb{A}^n/\Sigma_n$ to $\mathcal{M}$, and one can similarly descend the map $(\mathbb{A}^2)^n \times_{\mathbb{A}^2/\Sigma_n} \text{Hilb}^n \to \text{Hilb}^n \mathbb{A}^2$ to a surjection $\mathcal{D} \to \mathcal{N}$ (tautologically letting $\mathcal{D}$ be the stack of maps $C \to S$ which are étale-locally pulled back from $(\mathbb{A}^2)^n \times_{\mathbb{A}^2/\Sigma_n} \text{Hilb}^n \to \text{Hilb}^n \mathbb{A}^2$). The map $\mathcal{D} \to \mathcal{N}$ is not flat, and I expect that it is inferior to $C \to \mathcal{M}$ in other ways as well. Still, I believe that computing $H^*(\mathcal{N}, \mathbb{Z})$ is a realistic goal.
Naturality of the Puppe sequence

One of the obstructions to deducing $H^*(\mathcal{M}, \mathbb{Z})$ from $H^*(\mathcal{C}, \mathbb{Z})$ is that the connecting triangle is not natural for non-smooth maps. This makes it difficult to understand $H^*(\mathcal{M}, \mathbb{Z}) \rightarrow H^*(\mathcal{C}, \mathbb{Z})$ by working with one stratum at a time.

More explicitly, suppose that $Z' \hookrightarrow X'$ is a closed immersion with open complement $U'$, $\pi : X' \rightarrow X$ is a non-smooth map, $Z$ is the image of $Z'$ in $X$, and $U$ is the complement of $Z$:

$$
\begin{array}{ccc}
Z' & \hookrightarrow & X' \\
\downarrow & & \downarrow \\
Z & \hookrightarrow & X \\
\downarrow & & \downarrow \\
& & U
\end{array}
$$

Suppose further that $Z$ and $Z'$ are both smooth and pure of codimension $r$. Then the connecting triangle gives two long exact sequences, with maps between them making the diagram below commute:

$$
\begin{array}{cccccc}
\ldots & H^{i+r}(Z', \mathbb{Z}) & \longrightarrow & H^i(X', \mathbb{Z}) & \longrightarrow & H^i(U', \mathbb{Z}) & \longrightarrow & H^{i-1+r}(Z', \mathbb{Z}) & \longrightarrow & \ldots \\
\phi & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\ldots & H^{1+i+r}(Z, \mathbb{Z}) & \longrightarrow & H^i(X, \mathbb{Z}) & \longrightarrow & H^i(U, \mathbb{Z}) & \longrightarrow & H^{1+i-1+r}(Z, \mathbb{Z}) & \longrightarrow & \ldots
\end{array}
$$

Unfortunately $\phi$ is not the natural map induced by $Z' \rightarrow Z$. This makes it hard to understand the cohomology of a non-smooth map between stratified spaces, although in good circumstances you can compute $\phi$ using methods from homology or intersection theory.

The connecting triangle involves manipulations of sheaves of abelian groups which can seem a bit mystical. Luckily, there is a much more concrete version of these facts in terms of Thom spaces. The Thom space $Th(E)$ of a vector bundle $E$ on $S$ is the topological space $Th(E) = E/(E - S)$, where $S$ is viewed as a subspace of $E$ via the zero section. If $N_Z$ is the normal bundle of $Z$, the purity theorem says that $Th(E)$ is weakly equivalent to $X/U$. Diagram 1 can be obtained by applying $H^*(-, \mathbb{Z})$ to the diagram of Puppe sequences

$$
\begin{array}{cccccc}
U' & \longrightarrow & X' & \longrightarrow & Th(Z') & \longrightarrow & \Sigma U' & \longrightarrow & \Sigma X' & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
U & \longrightarrow & X & \longrightarrow & Th(Z') & \longrightarrow & \Sigma U & \longrightarrow & \Sigma X & \longrightarrow & \ldots
\end{array}
$$

Understanding $\phi$ reduces to understanding $\Phi$. The problem is that $\Phi$ is not induced by the natural map $N_{Z'} \rightarrow N_Z$ unless $X \rightarrow X'$ is smooth. Still, it is tempting to try to find some algebraic but nonlinear map $\Psi : N_{Z'} \rightarrow N_Z$ that induces $\Phi$. If it were explicit enough, $\Psi$ would make $\Phi$ and $\phi$ more transparent.

If there exists a $\mathbb{G}_m$ action on $X'$ and $X$ which makes $\pi : X' \rightarrow X$ is equivariant, fixes $Z$ and $Z'$, and acts with degree 1 on $N_Z$, then there is a natural candidate for $\Psi$. (These conditions hold Zariski-locally for our original case of interest $\mathcal{C} \rightarrow \mathcal{M}$, and for reasons specific to that case, Zariski-locally is enough). To define $\Psi$ we need to define an algebra morphism $\pi^*\text{Sym}^*N_Z^* \rightarrow \text{Sym}^*N_{Z'}^*$. Using the $\mathbb{G}_m$ action we can decompose $\pi^*N_Z^*$ by degree, and we can then send the degree-$d$ sections of $\pi^*N_Z^*$ to $\text{Sym}^dN_{Z'}^*$. If $X' \rightarrow X$ is a map of $\mathbb{G}_m$ representations and $Z', Z$ are the origin of $X'$ and $X$ respectively, then one can check that $\Psi$, as defined above, induces $\Phi$. Does this work in general?

The best setting for this question is probably $\mathbb{A}^1$ homotopy theory. Morel and Voevodsky prove the purity theorem in $\mathbb{A}^1$ homotopy theory by reducing to case $X = \mathbb{A}^n$, $Z = \mathbb{A}^{n-r}$, and I intend to mimic that method. However, where Morel and Voevodsky are trying to show that a map is an $\mathbb{A}^1$ weak equivalence, we are trying to show that two maps (namely $\Phi$ and the map induced by $\Psi$) are identical. Such local arguments in homotopy categories are delicate, and this problem seems to be more than just an exercise.