Bilipschitz equivalence is not equivalent to quasi-isometric equivalence for finitely generated groups

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Abstract

We show that certain lamplighter groups that are quasi-isometric to each other are not bilipschitz equivalent. This gives a positive answer to a question in [H] page 107.

A *quasi-isometric equivalence* between metric spaces is a map $f : X \to Y$ such that for some $K, C > 0$

$$-C + \frac{1}{K} d(x, y) \leq d(f(x), f(y)) \leq K d(x, y) + C$$

for all $x, y \in X$ and such that $nbhd_C(f(X)) = Y$. This is a generalization of the more common notion of a *bilipschitz equivalence*: a bijection between metric spaces that satisfies for some K

$$\frac{1}{K} d(x, y) \leq d(f(x), f(y)) \leq K d(x, y).$$

A natural question to ask is for which classes of metric spaces are these two notions equivalent. In [BK1] and [M], Burago-Kleiner and McMullen gave examples of a separated nets in $\mathbb{R}^2$ that are not bilipschitz equivalent to the integer lattice (but all nets are quasi-isometric). Our interest is in the class of finitely generated groups equipped with word metrics. For a finitely generated group Γ a choice of generating set S determines a Cayley graph Γ_S with metric d_S. The metric d_S depends on S but for any given group all Cayley graphs are bilipschitz equivalent. The examples in [BK1] and [M] are not Cayley graphs of finitely generated groups. In this paper we prove the following Theorem:

Theorem 1 *Let F and G be finite groups with $|F| = n$ and $|G| = n^k$ where $k > 1$. Then there does not exist a bijective quasi-isometry between the lamplighter groups $G \wr \mathbb{Z}$ and $F \wr \mathbb{Z}$ if k is not a product of prime factors appearing in n.*

For discrete spaces, a bijective quasi-isometry is the same as a bilipschitz map. In [Wh], Whyte proved that for *nonamenable* finitely generated groups any quasi-isometry is a bounded distance from a bijective quasi-isometry so our examples are necessarily *amenable*. Recall that a *Følner* sequence in a discrete metric space is a sequence of finite sets S_i such that for all $r > 0$

$$\lim_{i \to \infty} \frac{|\partial_r S_i|}{|S_i|} \to 0$$

where $\partial_r S_i$ denotes all points that are either not in S_i but at a distance of at most r from S_i or points in S_i at distance at most r from points not in S_i. A group is said to be *nonamenable* if it does not admit a Følner sequence and *amenable* otherwise. Using uniformly finite homology, one can check when a map between (uniformly discrete bounded geometry) spaces is bounded distance from a bijection (see [Wh] or Section 4 for details). The obstruction for a quasi-isometry to be a bounded distance from a bijection can only come from the existence of a Følner sequence. Since nonamenable groups do not have Følner sequences any quasi-isometry is a bounded distance from a bijection.

With respect to a certain generating set, the Cayley graph of the lamplighter group $F \wr \mathbb{Z}$ where $|F| = n$ is the Diestel Leader graph $DL(n, n)$. We will describe the geometry of Diestel Leader graphs and construct explicit Følner sequences in these graphs in Section 1. The main resource we have for analyzing quasi-isometries of Diestel Leader graphs is the following theorem of Eskin-Fisher-Whyte:

Theorem 2 *[EFW1] Any quasi-isometry $\varphi : DL(n, n) \to DL(n, n)$ is a bounded distance from a "standard" map of the form $(x, y, t) \mapsto (\varphi_l(x), \varphi_u(y), t)$ where $x, y \in \mathbb{Q}_n$ and $t \in \mathbb{R}$ and φ_l, φ_u are bilipschitz.*

For an explanation of the coordinatization of $DL(n, n)$ as $\mathbb{Q}_n \times \mathbb{Q}_n \times \mathbb{R}$ see Section 5.

Now since any two finite groups G and G' that have the same order give rise to the same Diestel Leader graph we can restrict our attention to $G = F^k$ where F^k is the direct product of k copies of F. The group $F^k \wr \mathbb{Z}$ appears as a finite index subgroup of $F \wr \mathbb{Z}$ of index k (see Section 1.1). By [D] we know that finite index subgroup inclusion $i : F^k \wr \mathbb{Z} \to F \wr \mathbb{Z}$ is not a bounded distance from a bijection. However, as pointed out in [D], if we were able to find a quasi-isometry

$$\varphi : F^k \wr \mathbb{Z} \to F^k \wr \mathbb{Z}$$

for which $|\varphi^{-1}(p)| = k$, also known as a k-to-1 quasi-isometry, then $i \circ \varphi : F^k \wr \mathbb{Z} \to F \wr \mathbb{Z}$ would be a bounded distance from a bijective quasi-isometry. We claim, first, that no such φ exists if k is not a product of prime powers appearing in n, and second, that any bijective quasi-isometry φ' between $F^k \wr \mathbb{Z}$ and $F \wr \mathbb{Z}$ arises in this way, i.e. φ' is a bounded distance from $i \circ \varphi$ for some φ.

The second claim is a simple consequence of the geometry of the graph $DL(n, n)$ (see Section 1.1). The proof of the first claim proceeds by contradiction. By Theorem 2 above, the k-to-1 map φ has the form $(\varphi_l, \varphi_u, id)$. Combining a theorem of Cooper [FM1] on the structure of bilipschitz maps of $\mathbb{Q}_n$ with a key observation of Juan Souto we are able to replace φ with a map $\bar{\varphi}$ that is still k-to-1 but that is now a bounded distance from a map $(\bar{\varphi}_l, \bar{\varphi}_u, Id)$ where $\bar{\varphi}_l, \bar{\varphi}_u$ are now *measure linear*, i.e. on the level of measure, φ_l, φ_u scale sets by fixed amounts λ_l, λ_u. This map $\bar{\varphi}$ is constructed in Section 3.2.

In Proposition 8 we are able to argue that λ_l, λ_u are products of powers of primes appearing in n. Finally in Section 5 we construct an explicit quasi-isometry ψ that is bounded distance from $\bar{\varphi}$ but that on a sequence of Følner sets is approximately $1/\lambda_l\lambda_u$-to-1. Using a Theorem of Whyte [Wh] we conclude that this is only possible if $1/\lambda_l\lambda_u = k$.

As a corollary of the proof we also answer a question of Burago-Kleiner [BK2]:

Corollary 3 *If $G \wr \mathbb{Z}$ is a lamplighter group with $|G| = n$ then $(G \wr \mathbb{Z}) \times F$ is not bilipschitz equivalent to $G \wr \mathbb{Z}$ if $|F|$ is not a product of primes appearing in n.*

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1 Geometry of $DL(n, n)$

There are now many places in the literature that describe the connection between the group $F \wr \mathbb{Z}$ and the graph $DL(n, n)$. See for example [TW, Wo]. We will freely borrow from these treatments. Understanding the group $F \wr \mathbb{Z}$ itself is not as important for us as understanding the geometry of $DL(n, n)$ so we will focus our energy on describing this graph.

Let T_n be a regular directed $n + 1$ valent infinite tree where each vertex has n incoming edges and one out going edge. Fixing a basepoint $p \in T_{n+1}$ we define a "height map" $h : T_{n+1} \to \mathbb{R}$ by setting $h(p) = 0$ and then mapping each coherently oriented line in T_{n+1} isometrically onto $\mathbb{R}$ with vertices mapping onto $\mathbb{Z}$. We define

$$DL(n, n) = \{(p, q) \in T_{n+1} \times T_{n+1} \mid h(p) + h(q) = 0\}.$$

On $DL(n, n)$ we can also define a height map $ht : DL(n, n) \to \mathbb{R}$ by setting $ht(p, q) = h(p) = -h(q)$.

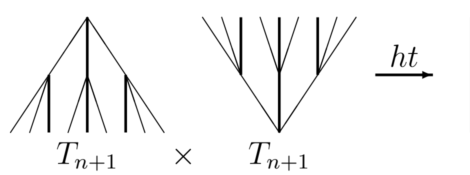

Figure 1: a portion of DL(3,3)

By drawing the second tree "upside down" in Figure 1, we can represent a vertex in $DL(3, 3)$ by a pairs of vertices, one in each tree, that have the same height. Likewise, a pair of edges at the same height represents an edge in $DL(3, 3)$. A portion of the actual graph can be seen in Figure 2. Both figures represent a connected component of $ht^{-1}([a, b])$ where $a, b \in \mathbb{Z}$ with $b - a = 2$.

Definition 1 (box) *We call a connected component of $ht^{-1}([a, b])$ where $a, b \in \mathbb{Z}$ and $b-a = H$ a box of height H.*

Note that a box S_H of height H has volume and boundary

$$|S_H| = (H + 1)n^H, \quad |\partial S_H| \sim 2 \cdot n^H.$$

Therefore, for any increasing sequence H_i, the sequence $\{S_{H_i}\}$ is a Følner sequence.

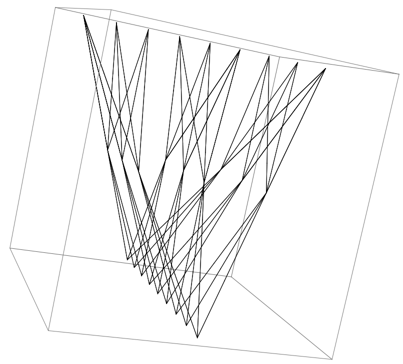

Figure 2: a portion of DL(3,3)

1.1 Subgroups of index k in $F \wr \mathbb{Z}$

As mentioned in the introduction, the group $F^k \wr \mathbb{Z}$ appears as a subgroup of index k in $F \wr \mathbb{Z}$. To see this, let $\rho : F \wr \mathbb{Z} \to \mathbb{Z}$ be the homomorphism that is projection onto the second factor. Then

$$\rho^{-1}(k\mathbb{Z}) \simeq F^k \wr \mathbb{Z}.$$

From a geometric point of view this inclusion allows us to see $DL(n^k, n^k)$ sitting inside of $DL(n, n)$. The map ρ is simply the height map ht. In figure 3, the egdes of $DL(3^2, 3^2)$ are distorted to show how $DL(3^2, 3^2)$ maps into $DL(3, 3)$.

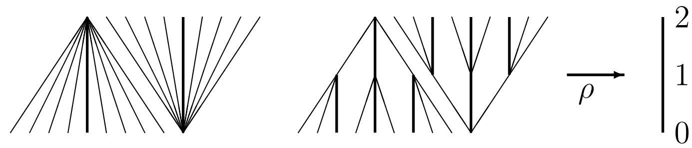

Figure 3: $DL(3^2, 3^2)$ includes into $DL(3, 3)$.

Definition 2 *For any k, we can define a map*

$$up : DL(n, n) \to i(DL(n^k, n^k))$$

by sending (p, q) to some point (p', q') where

- $ht(p') = ht(q') \in k\mathbb{Z}$,
- $ht(q') = ht(p') \geq ht(p) = ht(q)$
- $d_{T_{n+1}}(p, p') = d_{T_{n+1}}(q, q') < k$.

This map fixes points in $i(DL(n^k, n^k))$ and maps other points "up" to the closest level above them in $i(DL(n^k, n^k))$. Of course this *up* map is not unique but it can always be arranged so that for all $x \in i(DL(n^k, n^k))$ the preimage has $|up^{-1}(x)| = k$. Note that the *up* map is a bounded distance from the identity map on $DL(n, n)$.

Lemma 4 *Any bijective quasi-isometry $\varphi' : DL(n^k, n^k) \to DL(n, n)$ is a bounded distance from a map of the form*

$$i \circ \varphi$$

where $\varphi : DL(n^k, n^k) \to DL(n^k, n^k)$ is a k-to-1 quasi-isometry.

Proof. Let $\varphi = i^{-1} \circ up \circ \varphi'$ then φ is a k-to-1 map and $i \circ \varphi = up \circ \varphi'$ is a bounded distance from φ'. Each of φ', i and up are quasi-isometries so φ is also a quasi-isometry. ■

2 Boundaries

The material in this section can also be found in [EFW1] or [FM1] but we reproduce it here for the convenience of the reader.

2.1 Tree Boundaries

The oriented tree T_{n+1} has a special class of bi-infinite geodesics: the vertical geodesics. (Recall that each vertex in T_{n+1} has n incoming and one outgoing edges.) These are lines in T_{n+1} that are coherently oriented. Equivalently they are the lines that project isometrically onto $\mathbb{R}$ under the height map. Similarly, we can define vertical geodesic rays as sub-rays of vertical geodesics. We define the *tree boundary* to be equivalence classes of vertical geodesic rays. This coincides with the usual notion of tree boundary but our definition allows us to partition the boundary as

$$\partial_l T_{n+1} \cup \{\infty\}$$

where ∞ corresponds to the one class of positively oriented vertical rays. We will simply write ∂T_{n+1} to mean $\partial_l T_{n+1}$.

We can identify ∂T_{n+1} with $\mathbb{Q}_n$ by the following procedure. First, choose a coherently oriented line in T_{n+1} and label each edge on this line with 0. Then for each vertex assign labels $0, \ldots, n-1$ to incoming edges. Now each coherently oriented line ℓ in T_{n+1} (vertical geodesic) defines an element of $\mathbb{Q}_n$ by

$$\ell \mapsto \sum a_i n^{-i}$$

where a_i is the label of the edge in ℓ going from height i to $i+1$. Since each line eventually has all edges labeled 0 this is indeed a map to $\mathbb{Q}_n$. Likewise, each element of $\mathbb{Q}_n$ defines a line in T_{n+1}. The usual metric on $\mathbb{Q}_n$ has the following geometric interpretation. Consider two points ξ, ξ' on the boundary and two vertical geodesics γ, γ' emanating from ξ and ξ'. These two geodesics meet for the first time at some height I. This is precisely the index below which the two series for $\xi, \xi' \in \mathbb{Q}_n$ coincide. Hence $d_{\mathbb{Q}_n}(\xi, \xi') = n^I$.

2.1.1 Clones and measure

Definition 3 (clone) *Given a point $p \in T_{n+1}$ we define the* shadow *of p in ∂T_{n+1} to be all $\xi \in \partial T_{n+1}$ that can be reached by vertical geodesics passing through p. Any subset C of $\mathbb{Q}_n$ that can be defined in this way will be called a clone.*

We also have a natural measure on $\mathbb{Q}_n$ (Hausdorff measure) but since we will only be concerned with finite unions of clones we simply define for any clone C

$$\mu(C) := diam(C)$$

and for any finite union of disjoint clones $C = \sqcup A_i$

$$\mu(C) := \sum \mu(A_i).$$

Note that if C is the shadow of a point p then $\mu(C) = n^{h(p)}$.

2.2 $DL(n, n)$ boundaries

As in the tree case, we have an orientation on $DL(n, n)$ and a notion of vertical geodesic rays. We define two boundaries for $DL(n, n)$. The lower boundary, $\partial_l DL(n, n)$, is defined to be equivalence classes of downward oriented vertical geodesic rays whereas the upper boundary, $\partial_u DL(n, n)$, is the equivalence classes of upward oriented vertical geodesic rays. As pointed out in [EFW1], we can identify $\partial_l DL(n, n) \simeq \partial T_{n+1} \simeq \partial_u DL(n, n) \simeq \mathbb{Q}_n$.

Given a point $p \in DL(n, n)$ we can now define two different shadows, the lower shadow C_l and the upper shadow C_u. Both of these shadows are clones in $\mathbb{Q}_n$ with the property that $diam(C_u) = 1/diam(C_l)$ or, in terms of measure, $\mu(C_l)\mu(C_u) = 1$.

2.3 Height respecting quasi-isometries

A height respecting quasi-isometry of T_{n+1} or $DL(n, n)$ is a quasi-isometry that permutes level sets of the height function up to bounded distance such that the induced map on height is a translation. Theorem 2 says that all quasi-isometries of $DL(n, n)$ are height respecting. It is a straightforward computation to see that height respecting quasi-isometries induce bilipschitz maps of ∂_l and ∂_u (see [EFW1]). Likewise, height respecting isometries induce *similarities*, maps that scale distance by a fixed factor, on each of the boundaries.

3 Structure of bilipschitz maps of $\mathbb{Q}_n$

3.1 Preliminaries

In this section we use a theorem of Cooper to analyze the structure of surjective bilipschitz maps of $\mathbb{Q}_n$. The definitions and theorems used in this section can be found in [FM1].

Definition 4 *A map* $\varphi : \mathbb{Q}_n \to \mathbb{Q}_n$ *is said to be* measure linear *on a clone* $C \subset \mathbb{Q}_n$ *if there exists some* λ *such that for all* $A \subset C$

$$\frac{\mu(\varphi(A))}{\mu(A)} = \lambda.$$

The following theorem is a special case of Theorem 10.6 in [FM1].

Theorem 10.6 (Cooper) Suppose that C, C' are clones in $\mathbb{Q}_n$ and that φ is a bilipschitz map of C onto a clopen subset of C'. Then there is a clopen A in C such that the restriction $\varphi|_A$ of φ to A is measure linear.

As pointed out by Cooper in [FM1], a clopen in $C \subset \mathbb{Q}_n$ is a finite union of clones. To prove Proposition 5 below we simply need to show that for a surjective bilipschitz map $\varphi : \mathbb{Q}_n \to \mathbb{Q}_n$ the image of a clone is a finite union of clones. We can then apply Cooper's theorem.

Proposition 5 *Given a bilipschitz map* $\varphi : \mathbb{Q}_n \to \mathbb{Q}_n$ *there is a clone* $A \subset \mathbb{Q}_n$ *on which* $\varphi|_A$ *is measure linear.*

Proof. We show that for any clone C the image $\varphi(C)$ is a finite union of clones.

Claim $B \subset \mathbb{Q}_n$ is a finite union of clones if and only if for some $\epsilon > 0$

$$sep(B) := \inf\{d_{\mathbb{Q}_n}(x, y) \mid x \in B, y \notin B\} > \epsilon.$$

Proof of Claim. If B is a finite union of clones then $sep(B) > diam(D_{min})$ where D_{min} is the smallest clone in the union. Conversely if $sep(B) > \epsilon$ then consider clones D with size $diam(D) < \epsilon$. If $D \cap B \neq \emptyset$ then $D \subset B$ otherwise there would be $x, y \in D$ with $x \in B, y \notin B$ with $d(x, y) < diam(D) < \epsilon$ contradicting that $sep(B) > \epsilon$.

Now given $y \notin \varphi(C)$ there is some $z \notin C$ such that $\varphi(z) = y$. Since C is a clone, there exists ϵ_C such that $d(z, C) > \epsilon_C$ for all $z \notin C$. Then, since φ is bilipschitz, $d(y, \varphi(C)) \geq \frac{1}{K}\epsilon_C$. In particular $sep(\varphi(C)) > 1/K\epsilon_C$. This implies that $\varphi|_C$ maps onto a clopen of $C' \subset \mathbb{Q}_n$. Now we can apply Cooper's Theorem. ∎

3.2 Zooming

The following proposition was suggested by Souto.

Proposition 6 *If* $\varphi : \mathbb{Q}_n \to \mathbb{Q}_n$ *is a bilipschitz map then there exists a sequence* φ_i *of similarities of* $\mathbb{Q}_n$ *such that*

$$\lim_{i\to\infty} \varphi_i^{-1}\varphi\varphi_i = \bar{\varphi}$$

and $\bar{\varphi}$ *is measure linear on all of* $\mathbb{Q}_n$.

Proof. Since φ is bilipschitz then by Proposition 5 there is a clone $A \subset \mathbb{Q}_n$ such that $\varphi|_A$ is measure linear. We identify $\mathbb{Q}_n \simeq \partial T_{n+1}$ and bilipschitz maps (similarities) of $\mathbb{Q}_n$ with height respecting quasi-isometries (isometries) of T_{n+1}. In particular if φ is a quasi-isometry of T_{n+1} then we also write φ for the induced boundary map.

Let $p \in T_{n+1}$ and φ_i a sequence of isometries of T_{n+1} such that

$$\varphi_i(p) \to x \in A$$

and such that there is some vertical geodesic in $\ell \subset T_{n+1}$ and $M \geq 0$ such that $d(\varphi_i(p), \ell) \leq M$. We can pick such φ_i since the group of height-respecting isometries acts cocompactly on T_{n+1}. Then, up to composing φ with a height respecting isometry, the set $\{(\varphi_i^{-1}\varphi\varphi_i)(p)\}$ is bounded in T_{n+1} and so the quasi-isometries $\varphi_i^{-1}\varphi\varphi_i$ converge to some quasi-isometry $\bar{\varphi}$. This convergence can also be seen on the boundary.

Now for any clone $C \subset \mathbb{Q}_n$ there is some j_C such that if $i > j_C$ then $C \subset \varphi_i^{-1}(A)$ so that $\varphi_i^{-1}\varphi\varphi_i$ is measure linear on C if $i > j_C$. By Lemma 7 below, there is some i_C such that for $i > i_C$

$$\varphi_i(C) = \varphi(C).$$

Therefore φ is measure linear on all of $\mathbb{Q}_n$. ■

Lemma 7 *If $\varphi_i : \mathbb{Q}_n \to \mathbb{Q}_n$ is a sequence of bilipschitz maps that converge to φ then for all clones $C \subset \mathbb{Q}_n$ there exists i_C such that $\varphi_i(C) = \varphi(C)$ for $i \geq i_C$.*

Proof. Since $\varphi_i \to \varphi$, there exists i_ϵ such that for all $x \in C$ we have $d(\varphi(x), \varphi_i(x)) \leq \epsilon$, and therefore $\varphi_i(x) \in \varphi(C)$. Suppose $y_i \in \varphi(C) \setminus \varphi_i(C)$ then $y_i \to y$. But

$$y = \varphi(x) = \lim_{i\to\infty} \varphi_i(x)$$

Since $d(y_i, \varphi_i(x)) \to 0$ then by the same argument we have that $y_i \in \varphi(C)$ for $i > i_C$. ■

Proposition 8 *A map $\bar{\varphi} : \mathbb{Q}_n \to \mathbb{Q}_n$ that is onto and measure linear on all of $\mathbb{Q}_n$ has measure linear constant*

$$\lambda = r_1^{j_1} \dots r_i^{j_i}$$

where $j_l \in \mathbb{Z}$ and r_l is prime divisor of n.

Proof. Pick any $C \subset \mathbb{Q}_n$. Then $\bar{\varphi}(C)$ is a finite union of clones (since $\bar{\varphi}$ is bilipschitz). Of course any finite union of clones can also be written as a finite union of disjoint clones. Each clone B in this disjoint union has $\mu(B) = diam(B) = n^i$ for some i. Therefore λ is a finite sum of powers of n. Now $\bar{\varphi}$ has an inverse $\bar{\varphi}^{-1}$ that is also measure linear with measure linear constant $1/\lambda$. So both λ and $1/\lambda$ must be finite sums of powers n. The only possibilities are $\lambda = r_1^{j_1} \dots r_i^{j_i}$ since λ, when written as a fraction, must have denominator n^l for some $l \in \mathbb{N}$. ■

Corollary 9 *If $\varphi : DL(n,n) \to DL(n,n)$ is a quasi-isometry then there exists a quasi-isometry $\bar{\varphi}$ such that $\bar{\varphi}_l$ and $\bar{\varphi}_u$ are both measure linear on all of $\mathbb{Q}_n$.*

Proof. We modify the proof of Theorem 6 and repeat the zooming argument twice. The map φ has lower boundary map φ_l that is bilipschitz. Let A_l be a clone on which φ_l is measure linear as provided by Proposition 5. Pick φ_i to be isometries of $DL(n,n)$ such that $\varphi_i(p) \to x \in A_l$ and such that $\varphi_i(p)$ stays a bounded distance from some vertical geodesic in $DL(n,n)$. Then, by arguments similar to those in Theorem 6, $\varphi_i^{-1}\varphi\varphi_i \to \hat{\varphi}$ where $\hat{\varphi}_l$ has boundary maps $\hat{\varphi}_l$ and $\hat{\varphi}_u$ with $\hat{\varphi}_l$ measure linear on all of $\mathbb{Q}_n$. Now $\hat{\varphi}$ has upper boundary map $\hat{\varphi}_u$ which is bilipschitz and hence is measure linear on some clone $\hat{A}_u$. Repeating the above procedure with a sequence of isometries $\hat{\varphi}_i$ that now zoom into a point $y \in \hat{A}_u$ we get a new map $\bar{\varphi}$ that has boundary maps $\bar{\varphi}_l$ and $\bar{\varphi}_u$. The upper boundary map $\bar{\varphi}_u$ is measure linear by construction and the lower boundary map $\bar{\varphi}_l$ is also measure linear since the lower boundary maps of $\hat{\varphi}_i$ were all similarities. ■

Proposition 10 *If φ is an m-to-1 quasi-isometry then $\bar{\varphi}$ as constructed in Corollary 9 is bounded distance from an m-to-1 quasi-isometry.*

Proof. Recall that

$$\bar{\varphi} = \lim_{i\to\infty} \varphi_i^{-1}\varphi\varphi_i$$

where $\varphi_i, \varphi_i^{-1}$ are isometries and hence 1-to-1 maps. The composition of an m-to-1 map with a 1-to-1 map is again an m-to-1 map and a limit of a sequence of m-to-1 maps is also an m-to-1 map. ■

4 Uniformly finite homology

Uniformly finite homology was first introduced in [BW1] but we do not need the explicit construction here so we refer the reader to [BW1, BW2] for more details. Here we will only introduce the zeroth uniformly finite homology group.

Definition 5 *A metric space X is a* uniformly discrete bounded geometry *(UDBG) space if there exists $\epsilon > 0$ such that $d(x,y) > \epsilon$ for all $x \neq y$ and for all $r > 0$ there is a bound M_r on the size of any r ball.*

For a UDBG space X, let $C_0^{uf}(X)$ denote the vector space of infinite formal sums of the form

$$c = \sum_{x\in X} a_x x \quad (a_x \in \mathbb{Z})$$

for which there exists $M_c > 0$ such that $|a_x| < M_c$ for all $x \in X$. Let $C_1^{uf}(X)$ denote the vector space of infinite formal sums of the form

$$c = \sum_{x,y \in X} a_{(x,y)}(x,y)$$

for which there exists $M_c, R_c > 0$ such that $|a_{(x,y)}| < M_c$ and $a_{(x,y)} = 0$ if $d(x,y) > R_c$. The boundary map is defined by setting

$$\begin{array}{rcl} \partial : C_1^{uf}(X) & \to & C_0^{uf}(X) \\ (x,y) & \mapsto & y - x \end{array}$$

and extending by linearity. The zeroth homology group is then

$$H_0^{uf}(X) = C_0^{uf}(X)/\partial(C_1^{uf}(X)).$$

All Cayley graphs are UDBG spaces. Futhermore, this homology group is a quasi-isometry invariant so if X and Y are quasi-isometric then $H_0^{uf}(X) \simeq H_0^{uf}(Y)$. In particular, for a given group, the uniformly finite homology group does not depend on generating set.

Definition 6 *Any subset $S \subset X$ defines a class $[S] \in H_0^{uf}(X)$, which is the class of the chain $\sum_{x \in S} x$. We call $[X]$ the* fundamental class *of X in $H_0^{uf}(X)$.*

Using uniformly finite homology, Whyte developed in [Wh] a test to determine when a quasi-isometry between UDBG spaces is a bounded distance from a bijection.

Theorem 11 *[Wh] Let $f : X \to Y$ be a quasi-isometry between UDBG-spaces. Then there exists a bijective map a bounded distance from f if and only if $f_*([X]) = [Y]$ where*

$$f_*([X]) = \left[\sum_{y \in f(X)} |f^{-1}(y)| y \right].$$

Another fact that we will use is that two maps f, g that are a bounded distance apart have

$$f_*([X]) = g_*([X]).$$

The following theorem allows us to check when a chain c represents the class of the zero chain in $H_0^{uf}(X)$. We use this theorem in Proposition 15.

Theorem 12 *[Wh] Let X be a UDBG-space, and $c = \sum_{x \in X} a_x x \in C_0^{uf}(X)$. Then we have $[c] = 0 \in H_0^{uf}(X)$ if and only if there exist an r such that for any Følner sequence $\{S_i\}$,*

$$\left| \sum_{x \in S_i} a_x \right| = O(|\partial_r S_i|).$$

Remark. If $f : X \to X$ is m-to-1 then $f_*([X]) = m[X]$. If f is a bounded distance from an m-to-1 map then we also have $f_*([X]) = m[X]$. In case m is not an integer we will use this condition as the definition.

5 From boundary maps to interior maps

In this section we construct an explicit quasi-isometry $\psi : DL(n,n) \to DL(n,n)$ from two boundary maps $\bar{\varphi}_l$ and $\bar{\varphi}_u$. First we need to relate $DL(n,n)$ to $\mathbb{Q}_n \times \mathbb{Q}_n \times \mathbb{R}$. For this purpose we define two maps:

$$\pi : \mathbb{Q}_n \times \mathbb{Q}_n \times \mathbb{R} \to DL(n,n),$$

$$\bar{\pi} : DL(n,n) \to \mathbb{Q}_n \times \mathbb{Q}_n \times \mathbb{R}.$$

The map π collapses sets of the form $A \times B \times \{t\}$, where A and B have diameters $n^{\lfloor t \rfloor}$ and $n^{\lceil -t \rceil}$ respectively, to a single point at height $\lfloor t \rfloor$. This is a map onto the vertices of $DL(n,n)$. (We could have also defined this map so that it also mapped onto edges but for our purposes this is sufficient.) The map $\bar{\pi}$ is chosen so that $\pi\bar{\pi} = id$. We can endow $\mathbb{Q}_n \times \mathbb{Q}_n \times \mathbb{R}$ with a

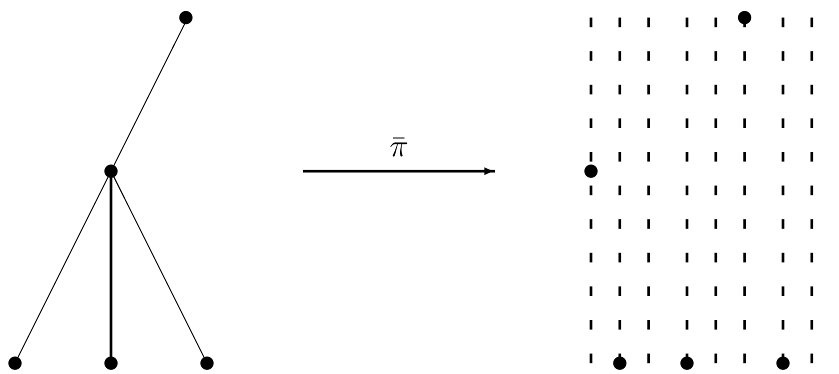

Figure 4: A possible choice for $\bar{\pi}$ shown only in projection onto T_3 and $\mathbb{Q}_3 \times \mathbb{R}$.

'Sol' like metric so that $\pi, \bar{\pi}$ are quasi-isometries. Specifically, for two points $(x,y,t), (x,y,t')$ that differ only in the $\mathbb{R}$ coordinate, we define their distance to be $|t-t'|$. For two points at the same height $(x,y,t), (x',y',t) \in \mathbb{Q}_n \times \mathbb{Q}_n \times \mathbb{R}$, we define

$$d((x,y,t),(x',y',t)) = n^{-t} d_{\mathbb{Q}_n}(x,x') + n^t d_{\mathbb{Q}_n}(y,y').$$

For points p, q at different heights, we consider all finite sequences $\{p_i\}$ connecting p and q where subsequent terms are either at the same height or differ only in the $\mathbb{R}$ coordinate and define the distance to be

$$d(p,q) = \inf \sum d(p_i, p_{i+1}).$$

Definition 7 *Given two bilipschitz maps $\bar{\varphi}_l, \bar{\varphi}_u$ of $\mathbb{Q}_n$, we define a quasi-isometry*

$$\psi : DL(n,n) \to DL(n,n)$$

by

$$\psi := \pi \circ (\bar{\varphi}_l \times \bar{\varphi}_u \times Id) \circ \bar{\pi}.$$

Lemma 13 *Let $C_l, C_u \subseteq \mathbb{Q}_n$ be two clones with $\mu(C_l)\mu(C_u) > 1$. Then the image*

$$S_{C_l,C_u} = \pi(C_l \times C_u \times [-\log_n(\mu(C_u)), \log_n(\mu(C_l))])$$

defines a box of height $H := |\log_n(\mu(C_u)\mu(C_l))|$ with $\mu(C_l)\mu(C_u)$ vertices at each height.

Proof. This follows from the definition of π. Recall that $|S_{C_l,C_u}| = (H+1)n^H$ while $|\partial S_{C_l,C_u}| = 2n^{H+1}$, so if we chose a sequence S_i of such boxes with height H_i increasing then $\{S_i\}$ is a Følner sequence. ■

Lemma 14 *Let S_{C_l,C_u} be a box as defined in Lemma 13 with height $H \gg \log_n K$. Then, for $r = \log_n K$,*

$$\frac{1}{\lambda_l\lambda_u}\left(|S_{C_l,C_u}| - |\partial_r S_{C_l,C_u}|\right) \leq \sum_{x \in S_{C_l,C_u}} |\psi^{-1}(x)| \leq \frac{1}{\lambda_l\lambda_u}|S_{C_l,C_u}| + K^2|\partial_r S_{C_l,C_u}|.$$

Proof. First we claim that since $\bar{\varphi}_l$ and $\bar{\varphi}_u$ are K-bilipschitz then

$$\bar{\varphi}_l^{-1}(C_l) = \sqcup A_i \quad \text{and} \quad \bar{\varphi}_u^{-1}(C_u) = \sqcup B_i$$

where A_i and B_i are clones of size $\mu(A_i) \geq \frac{1}{K}\mu(C_l)$ and $\mu(B_i) \geq \frac{1}{K}\mu(C_u)$. We already know that the preimages of $\bar{\varphi}_l$ and $\bar{\varphi}_u$ are unions of clones. To get an estimate on the size of these clones note that

$$n\mu(A_i) = sep(A_i) = sep(C_l) \geq \frac{1}{K}n\mu(C_l)$$

and similarily for B_i. Without loss of generality, we can assume that all A_i have the same size $\mu(A_i) = m_A$ and all the B_i have size $\mu(B_i) = m_B$. In particular, there are $\frac{\mu(C_l)}{\lambda_l m_A}$ many clones in $\bar{\varphi}_l^{-1}(C_l)$ and $\frac{\mu(C_u)}{\lambda_l m_B}$ many clones in $\bar{\varphi}_u^{-1}(C_u)$. This implies that for a fixed height t there are

$$\frac{\mu(C_l)}{\lambda_l m_A} \cdot \frac{\mu(C_u)}{\lambda_l m_B} = \frac{\mu(C_l)\mu(C_u)}{\lambda_l\lambda_u} \cdot \frac{1}{m_A m_B}$$

many sets of the form $A_i \times B_j \times \{t\}$ in the preimage of $\bar{\varphi}_l \times \bar{\varphi}_u \times id$. If $\log_n m_A \geq t \geq -\log_n m_B$ and $t \in \mathbb{Z}$ then $\pi(A_i \times B_j \times \{t\})$ contains $m_A m_B$ many vertices. More importantly

$$\bar{\pi}(DL(n,n)) \cap A_i \times B_j \times \{t\}$$

contains $m_A m_B$ many images of vertices. For t outside this range this may not be the case and depends on the choice of $\bar{\pi}$.

Now for height

$$t \in [-\log_n(\mu(C_u)) + \log_n K, \log_n(\mu(C_l)) - \log_n K])$$

we have

$$\frac{\mu(C_l)\mu(C_u)}{\lambda_l\lambda_u}\cdot\frac{1}{m_A m_B}\cdot m_A m_B=\frac{\mu(C_l)\mu(C_u)}{\lambda_l\lambda_u}$$

many vertices being mapped into S_{C_l,C_u} at height t since $\log_n m_A \leq \log_n(\mu(C_l)) - \log_n K$ and $-\log_n m_B \geq -\log_n(\mu(C_u)) + \log_n K$. But there are $\mu(C_l)\mu(C_u)$ many vertices at this level. Therefore, on average, there are $1/\lambda_l\lambda_u$ many vertices being mapped onto each vertex.

Finally, for t in the range

$$\log_n(\mu(C_l)) \geq t \geq \log_n(\mu(C_l)) - \log_n K \text{ or } \log_n(\mu(C_u)) \leq t \leq \log_n(\mu(C_u)) + \log_n K$$

it is possible, depending on the choice of $\bar{\pi}$, that no vertices are mapped to these heights or that as many as

$$diam(\bar{\varphi}_l^{-1}(C_l)) \cdot diam(\bar{\varphi}_u^{-1}(C_u))$$

vertices are mapped to each height. But $diam(\bar{\varphi}_l^{-1}(C_l)) \leq K\mu(C_l)$ and $diam(\bar{\varphi}_u^{-1}(C_u)) \leq K\mu(C_u)$ so there are at most $K^2\mu(C_l)\mu(C_u)$ vertices being mapped to each level or in other words on average at most K^2 to each vertex. Note also that these levels are all contained in $\partial_{\log_n K} S_i$. ■

Proposition 15 *Let $X = DL(n,n)$. If φ_l, φ_u are measure linear with constants λ_l, λ_u then*

$$\psi_*(X) \neq k[X]$$

if $k \neq 1/\lambda_l\lambda_u$.

Proof. We will use Whyte's Theorem 12. Let S_i be an increasing sequence of boxes in $DL(n,n)$ (a Følner sequence). Let c be the chain defined by

$$c = \sum_{x\in X} a_x x$$

where $a_x = |\psi^{-1}(x)| - k$. By Lemma 14 we have that for $r = \log_n K$

$$\left(\frac{1}{\lambda_l\lambda_u} - k\right)|S_i| - \frac{1}{\lambda_l\lambda_u}|\partial_r S_i| \leq \left|\sum_{x\in S_i} |\psi^{-1}(x)| - k\right| \leq \left(\frac{1}{\lambda_l\lambda_u} - k\right)|S_i| + K^2|\partial_r S_i|$$

so that unless $1/\lambda_l\lambda_u = k$ we have $|\sum_{x\in S_i} a_x|$ is not $O(|\partial_r S_i|)$ since $|S_i|$ is not $O(\partial_r S_i)$. ■

To conclude the proof of Theorem 1, we note that since ψ and $\bar{\varphi}$ have the same boundary maps they are a bounded distance apart. In particular,

$$\psi_*([X]) = \bar{\varphi}_*([X]).$$

Now if $\bar{\varphi}$ is k-to-1 then $\bar{\varphi}_*([X]) = k[X]$ but, as shown in Proposition 15, this is impossible.

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