NON-RECTIFIABLE DELONE SETS IN SOL AND OTHER SOLVABLE GROUPS

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Given a lattice $\Gamma \subset SOL$, we show that there is a coarsely dense subset $\mathcal{D} \subset \Gamma$ that is not biLipschitz equivalent to Γ. We also prove similar results for lattices in certain higher rank abelian-by-abelian groups and for the solvable Baumslag-Solitar groups.

Introduction

A Delone set in a metric space is a uniformly discrete, coarsely dense subset. This means that there exist positive constants C, D such that any two distinct points of this set are at distance $\geq D$, and any point in the space is at distance $\leq C$ of some point in this subset. Typical examples of Delone sets are co-compact lattices in Lie groups. In this work, we are interested in the biLipschitz equivalence of Delone subsets of certain solvable groups.

Let us first recall that there is a big difference between the non-amenable and amenable cases. In the former one, a general result of Whyte [W] establishes biLipschitz equivalence of any two Delone subsets of the same non-amenable finitely-generated group. (Actually, this holds in any non-amenable space provided some uniform condition on the geometry is satisfied.) For the amenable case, the situation is unclear. An easy argument shows that any two Delone subsets of $\mathbb{R}$ are biLipschitz equivalent. However, this is false in $\mathbb{R}^d$ for $d \geq 2$, as was shown by Burago-Kleiner [BK] and, independently, by McMullen [M]. Although not explicitly stated in these works, it turns out that their examples may be realized as Delone subsets of $\mathbb{Z}^d$ (see [CN]). This motivates the next general question:

Question. Let Γ be a finitely-generated infinite amenable group that is not a finite extension of $\mathbb{Z}$. Does Γ contain a Delone subset that is not biLipschitz equivalent to Γ itself?

We call such a Delone set *non-rectifiable*. The first non-rectifiable Delone sets in amenable groups other that $\mathbb{Z}^d$ were constructed by the first author in [D] by showing that any lamplighter group $\Gamma = F \wr \mathbb{Z}$ with $|F| < \infty$ contains finite-index subgroups that are not biLipschitz equivalent to Γ itself. These examples were further generalized in [DPT] to the so-called Diestel-Leader groups. All lattices in the three dimensional solvable Lie group SOL are biLipschitz equivalent so, in particular, for a given lattice Γ it cannot have non-rectifiable subgroups. In this work we show that, nevertheless, there do exist non-rectifiable Delone subsets in any lattice $\Gamma \subset SOL$. Additionally, we prove the same results for lattices in certain abelian-by-abelian solvable Lie groups, as well as for the solvable Baumslag-Solitar groups.

To construct the required subsets, we combine two ingredients. First, we use slight modifications of the combinatorial arguments of [CN], which correspond to discretized versions of the arguments in [BK, M] and perfectly fit in our context. Second, we crucially use the description of all quasi-isometies of the underlying group, which is available in each case. It is this last issue that unables us to treat the general case of the question above, so that the complete answer remains unclear.

Outline. The three different families of groups that we consider are each treated in separate sections; namely, Section 1 for lattices in SOL, Section 2 for lattices in certain higher-rank abelian-by-abelian groups, and Section 3 for solvable Baumslag-Solitar groups. The three theorems we prove are the following:

Theorem 1.7 *Any lattice in SOL contains a non-rectifiable Delone subset.*

Theorem 2.7 *Any lattice in a boundary one-dimensional, even-scaling, abelian-by-abelian Lie group contains non-rectifiable Delone subsets.*

(See Section 2 for a definition of *boundary one-dimensional, even-scaling,* abelian-by-abelian Lie groups.)

Theorem 3.8 *For all m, the solvable Baumslag-Solitar group*

$$BS(1,m) = \left\langle t, a \mid tat^{-1} = a^m \right\rangle$$

contains non-rectifiable Delone subsets.

Section 1, the case of lattices in SOL, serves as an introduction to the more general case of lattices in abelian-by-abelian Lie groups. (Indeed, SOL itself is a boundary one-dimensional, even-scaling, abelian-by-abelian Lie group). For any Delone set $\mathcal{D}$ and lattice Γ, a biLipschitz equivalence $\mathcal{D} \to \Gamma$ gives rise to a quasi-isometry of the ambient Lie group. Quasi-isometries of these Lie groups are understood by the work of Eskin-Fisher-Whyte in [EFW1, EFW2] and Peng in [P1, P2]. Their results imply that certain *box* Følner sets are almost preserved up to a quasi-isometry. This allows us to construct $\mathcal{D}$ by removing points from Γ in such a way that any quasi-isometry induced by a biLipschitz map $\mathcal{D} \to \Gamma$ would violate the conditions of the quasi-isometric rigidity theorems. Solvable Baumslag-Solitar groups are not lattices in any real Lie groups but their quasi-isometries have a similar structure to those of SOL. Again, we show that the images of certain special Følner sets are almost preserved under quasi-isometries, and this allows us to apply a similar proof to the one for lattices in SOL.

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1. Delone sets in SOL

The solvable Lie group $SOL = \mathbb{R}^2 \rtimes \mathbb{R}$ with the coordinates (x, y, t) has multiplication rule

$$(x_1, y_1, t_1) * (x_2, y_2, t_2) := (x_1 + e^{t_1} x_2,\ y_1 + e^{-t_1} y_2,\ t_1 + t_2)$$

and left-invariant metric

$$ds^2 = e^{-2t}dx^2 + e^{2t}dy^2 + dt^2,$$

which gives rise to a distance function quasi-isometric to

$$d\big((x_1,y_1,t_1),(x_2,y_2,t_2)\big) := e^{-\frac{t_1+t_2}{2}}|x_1-x_2| + e^{\frac{t_1+t_2}{2}}|y_1-y_2| + |t_1-t_2|.$$

A typical example of a lattice in SOL is given by $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}$, where the action of $\mathbb{Z}$ is given by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. (Indeed, any diagonalizable matrix with eigenvalues of norm $\neq 1$ will also give a lattice in SOL.)

In the next section we explain the construction of the Delone set $\mathcal{D}$ and then in the following sections we prove that it is indeed non-rectifiable.

1.1. **Tiling.**

In this section we construct the non-rectifiable Delone set $\mathcal{D}$. Let

$$S_N := [0,N) \times [0,1) \times [0,\log N)$$

be a subset of SOL. We can tile the larger set S_{N^2} by $2N$ translates of S_N by letting

$$S_{N^2} = \bigsqcup_{i,k} g_{k,i}S_N, \quad i = 1,2 \text{ and } k = 1,\dots,N,$$

where $g_{k,1} := ((k-1)N,0,0)$ and $g_{k,2} := (0, \frac{(k-1)}{N}, \log N)$. In other words, S_{N^2} is the disjoint union of

$$T_{k,1} := g_{k,1}S_N = [(k-1)N, kN) \times [0,1) \times [0,\log N)$$

and

$$T_{k,2} := g_{k,2}S_N = [0,N^2) \times [\frac{k-1}{N}, \frac{k}{N}) \times [\log N, 2\log N).$$

Now, inductively, for any $M := N^{2^m}$ we can tile S_M by $2^m N^{2^m-1}$ copies of S_N.

Fix Γ a lattice in SOL. We will construct our Delone set $\mathcal{D}$ inductively by fixing N_0 and picking four subsets $Q_j^0 \subset S_{N_0} \cap \Gamma$ for $j = 1,2,3,4$ and translating them by $g_{k,i}$ to define four new sets $Q_j^1 \subset S_{N_0^2} \cap \Gamma$. We have to be careful here since $g_{k,i}(S_N \cap \Gamma)$ is not necessarily a subset of Γ. However, since Γ is coarsely dense in SOL, there is always a $\gamma_{k,i} \in \Gamma$ with

$$d_{SOL}(\gamma_{k,i}, g_{k,i}) < C,$$

where C depends only on the coarse density of Γ in SOL. Therefore, we can first construct a Delone set $\hat{\mathcal{D}}$ that is at most distance C from a subset of Γ. Then by moving points in $\hat{\mathcal{D}}$ at most distance C we get the desired Delone set $\mathcal{D} \subset \Gamma$. We will make sure that the initial subsets Q_j^0 are chosen to be at least distance C from the boundary of S_{N_0} so that the natural bijections $\gamma_{k,i}x \mapsto g_{k,i}x$ between $\gamma_{k,i}Q_j^0$ and $g_{k,i}Q_j^0$ do not overlap.

Volumes. In order to make volume computations more feasible, we fix N_0 an even integer and pick n points from $\Gamma \cap S_{N_0}$. We denote this set Q_0^0 (i.e. $|Q_0^0| = n$). The Q_j^0 for $j = 1,2,3,4$ will all be subsets of Q_0^0. The number n will be chosen later and it will determine how large N_0 must be. As mentioned above, we pick these points to be at least distance C from the boundary of S_{N_0}. This ensures that for $g \in SOL$ and $\gamma \in \Gamma$ with $d_{SOL}(g,\gamma) < C$, we have

$$|Q_0^0| = |gQ_0^0| = |\gamma Q_0^0 \cap gS_{N_0}| = |\gamma Q_0^0|.$$

If we set $Q_0^1 = \bigsqcup_{i,k} g_{k,i} Q_0^0$ and then inductively define Q_0^m, we find that the volume of Q_0^m is given by

$$|Q_0^m| = 2^m N_0^{2^m - 1} |Q_0^0| = M \log_{N_0}(M) \frac{n}{N_0},$$

where $M = N_0^{2^m}$.

Basic tiles. We fix d_1, d_2, d_3, d_4 rational constants in $(0,1]$ that satisfy $d_1 < d_2$ as well as

$$d_3 = \frac{3d_1 - d_2}{2} \quad \text{and} \quad d_4 = \frac{3d_2 - d_1}{2}.$$

(For instance, pick $d_1 = 1/3, d_2 = 1/2, d_3 = 1/4, d_4 = 7/12$.) Notice that $d_3 < d_1 < d_2 < d_4$. We choose n so that $d_j n \in \mathbb{N}$ for each $j = 1, \dots, 4$. We define basic tiles $Q_j^0 \subset Q_0^0 \subset S_{N_0}$, $j = 1, \dots, 4$, with

$$|Q_j^0| = d_j |Q_0^0| = d_j n,$$

by arbitrarily picking the required number of points from Q_0^0.

Inductively defined tiles. Define Q_j^1 for $j = 1,2,3,4$ as follows: Consider

$$T_{k,1} := [(k-1)N_0, kN_0) \times [0,1) \times [0, \log N_0)$$

for $k = 1, \dots, N_0$, as before. If k is odd, set $Q_1^1|_{T_{k,1}} = Q_2^1|_{T_{k,1}} := g_{k,1} Q_1^0$, and if k is even, set $Q_1^1|_{T_{k,1}} = Q_2^1|_{T_{k,1}} := g_{k,1} Q_2^0$. (Here, as before, $g_{k,1} \in SOL$ is the element that takes S_{N_0} to $T_{k,1}$.)

Next, consider

$$T_{k,2} := [0, {N_0}^2) \times [\frac{k-1}{N_0}, \frac{k}{N_0}) \times [\log N_0, 2 \log N_0)$$

for $k = 1, \dots, N_0$. Then set $Q_1^1|_{T_{k,2}} := g_{k,2} Q_3^0$ and $Q_2^1|_{T_{k,2}} := g_{k,2} Q_4^0$.

Now we compute the volume of these sets:

$$\begin{aligned}
|Q_1^1| &= \frac{N_0}{2}|Q_1^0| + \frac{N_0}{2}|Q_2^0| + N_0|Q_3^0| \\
&= \frac{d_1}{2} N_0 |Q_0^0| + \frac{d_2}{2} N_0 |Q_0^0| + d_3 N_0 |Q_0^0| \\
&= \left(\frac{d_1}{2} + \frac{d_2}{2} + d_3 \right) \frac{1}{2} |Q_0^1| \\
&= d_1 |Q_0^1|,
\end{aligned}$$

and similarly,

$$\begin{aligned}
|Q_2^1| &= \frac{N_0}{2}|Q_1^0| + \frac{N_0}{2}|Q_2^0| + N_0|Q_4^0| \\
&= \left(\frac{d_1}{2} + \frac{d_2}{2} + d_4 \right) \frac{1}{2} |Q_0^1| \\
&= d_2 |Q_0^1|,
\end{aligned}$$

where the last equality in each case comes, respectively, from

$$\left(\frac{d_1}{2} + \frac{d_2}{2} + d_3 \right) \frac{1}{2} = d_1 \quad \text{and} \quad \left(\frac{d_1}{2} + \frac{d_2}{2} + d_4 \right) \frac{1}{2} = d_2.$$

To define Q_3^1 and Q_4^1 simply set $Q_3^1 := \bigsqcup_{i,k} g_{k,i} Q_3^0$ and $Q_4^1 := \bigsqcup_{i,k} g_{k,i} Q_4^0$, so that trivially $|Q_3^1| = d_3 |Q_0^1|$ and $|Q_4^1| = d_4 |Q_0^1|$.

Now, we repeat this process to inductively define Q_j^m for $j = 1, \dots, 4$, with

$$|Q_j^m| = d_j |Q_0^m|.$$

Next, we let $\hat{\mathcal{D}}$ be the union of all Q_j^1. This defines $\hat{\mathcal{D}}$ on $\Gamma \cap U$, where

$$U := \{(x, y, t) \mid x, t \text{ non negative, } y \in [0,1]\}.$$

We then extend $\hat{\mathcal{D}}$ to $SOL \smallsetminus U$ simply by taking $\hat{\mathcal{D}} = \Gamma$ therein. Finally, we let $\mathcal{D}$ be the subset of Γ obtained by translating points of $\hat{\mathcal{D}}$ a distance $\leq C$, as previously explained.

1.2. **Quasi-isometries.** Any K-biLipschitz map $f : \mathcal{D} \to \Gamma$ extends to a (K, C) quasi-isometry of SOL, where C depends on the coarse density constants of $\mathcal{D}$ and Γ. By Eskin-Fisher-Whyte [EFW1, EFW2], this quasi-isometry is, up to an isometry that permutes the first two factors, at bounded distance $A = A(K, C)$ from a "companion" quasi-isometry F of the form

$$F(x, y, t) = (f_1(x), f_2(y), t),$$

where f_1, f_2 are L-biLipschitz maps of $\mathbb{R}$. As long as f coarsely fixes the identity, the constant L depends only on K and C. Here we use the term "coarsely fixes" the identity to mean that it maps the identity at most distance C from the identity in the image. We can always arrange for f to coarsely fix the identity by composing f with left multiplication by a group element in Γ. We will also assume that f does not permute the first two factors since this does not change the arguments in the proof. Henceforth, companion quasi-isometries will be always chosen with these properties.

1.3. **Følner sets.** Recall that a *Følner sequence* in a finitely generated group Γ is a sequence of finite sets S_i such that for all $R > 0$

$$\lim_{i \to \infty} \frac{|\partial_R S_i|}{|S_i|} = 0$$

where

$$\partial_R S_i := \{x \in \Gamma \mid d_\Gamma(x, \Gamma \smallsetminus S_i) \leq R \text{ and } d_\Gamma(x, S_i) \leq R\}$$

and d_Γ is any metric quasi-isometric to a word metric on Γ. In a Lie group, the definition is similar but with finite sets replaced with compact sets and counting measure by volume (Haar measure). Although the next terminology is not very precise, any set that belongs to a Følner sequence will be called a *Følner set*.

In the next lemma, we show that in a Lie group G, any Følner set $S \subset G$ defines a Følner set $\bar{S} := S \cap \Gamma$ in Γ for any uniform lattice $\Gamma \subset G$. To make computations easier we chose d_Γ to be the restriction of the metric d_G on G. This allows us to compare $|S|$ with $|S \cap \Gamma|$ and $|\partial_R S|$ with $|\partial_R (S \cap \Gamma)|$. We also assume that $|G/\Gamma| = 1$.

Lemma 1.1. *Let $D := \operatorname{diam}(G/\Gamma)$. For any set S and any $R \gg D$, we have that*

$$\frac{|\partial_{R-2D} S|}{|S| + |\partial_D S|} \leq \frac{|\partial_R (S \cap \Gamma)|}{|S \cap \Gamma|} \leq \frac{|\partial_{R+D} S|}{|S| - |\partial_D S|}.$$

Proof. First we claim that for any S,

$$|S \smallsetminus \partial_D S| \le |S \cap \Gamma| \le |S \cup \partial_D S|. \tag{1}$$

To see this, let E be a fundamental domain, containing the identity, for the action of Γ on G (so that $|E| = 1$ and $\operatorname{diam}(E) = D$), and let $S' = (\Gamma \cap S) \cdot E$. Then $|S'| = |S \cap \Gamma|$ and

$$S \smallsetminus \partial_D S \subset S' \subset \partial_D S \cup S.$$

The second inclusion is clear since $\operatorname{diam}(E) = D$ and so for any $\gamma \in S \cap \Gamma$ we must have $\gamma \cdot E \subset \partial_D S \cup S$. The first inclusion follows from the fact that any $x \in S \smallsetminus S'$ must lie in some $\gamma \cdot E$, in which case $\gamma \in \partial_D S \smallsetminus S$, and since $d(\gamma, x) \le D$, we must have $x \in \partial_D S$.

This claim gives us

$$|S| - |\partial_D S| \le |S \cap \Gamma| \le |S| + |\partial_D S|.$$

Futhermore, for any $R \gg D$ we have that

$$|\partial_{R-2D} S| \le |\partial_{R-D} S \cap \Gamma| \le |\partial_R (S \cap \Gamma)| \le |\partial_R S \cap \Gamma| \le |\partial_{R+D} S|,$$

and therefore

$$\frac{|\partial_{R-2D} S|}{|S| + |\partial_D S|} \le \frac{|\partial_R (S \cap \Gamma)|}{|S \cap \Gamma|} \le \frac{|\partial_{R+D} S|}{|S| - |\partial_D S|},$$

as desired. $\square$

We now define a family of preferred Følner sets in SOL (and hence Γ). A *standard* Følner set in SOL is given by

$$U_r \times U_s \times [-\log s, \log r)$$

where U_r and U_s are intervals of length r and s respectively. Note that S_N defined in the previous section is a preferred Følner set with $r = N$ and $s = 1$. As long as $rs > 1$, the volume of this set is $rs \log(rs)$, and the volume of the R-boundary is bounded by $\bar{C} rs$, where $\bar{C}$ depends on R. For the case of S_N, this is proved in detail just after Corollary 2.4 in Section 2.

We also need to estimate the boundary volume ratio for images of standard Følner sets; namely sets of the form

$$U_{ar} \times U_{bs} \times [-\log s, \log r),$$

with $a, b > 0$. Any set of this form is also a Følner set as long as we fix a, b and let $rs \to \infty$. Again, this follows from Corollary 2.4 in the next section.

The following lemma will be applied with $G = SOL$ in this section and G a higher-rank abelian-by-abelian solvable Lie group in the next section.

Lemma 1.2. *For any Følner set $S \subset G$, suppose that $f : \mathcal{D} \to \Gamma$ and $F : G \to G$ are bounded distance A apart on $\mathcal{D}$. Then the following inequality holds:*

$$|F(S)| - |\partial_{A+D} F(S)| \le |f(\mathcal{D} \cap S)| \le |F(S)| + |\partial_{A+D} F(S)|.$$

Proof. Since f is bounded distance A from F, the most that we can gain or lose by replacing F with f comes from the A-boundary of the Følner set $F(S)$. Specifically,

$$\Gamma \cap (F(S) \smallsetminus \partial_A(F(S))) \subset f(\mathcal{D} \cap S) \subset (F(S) \cup \partial_A(F(S))) \cap \Gamma.$$

By (1), since $F(S) \cup \partial_A(F(S))$ is also a Følner set, this yields

$$\begin{aligned}
|(F(S) \cup \partial_A(F(S))) \cap \Gamma| &\leq |F(S) \cup \partial_A(F(S))| + |\partial_D(F(S) \cup \partial_{A+D}(F(S))| \\
&\leq |F(S)| + |\partial_{A+D}(F(S))|
\end{aligned}$$

and

$$\begin{aligned}
|(F(S) \setminus \partial_A(F(S))) \cap \Gamma| &\geq |F(S) \setminus \partial_A(F(S))| - |\partial_D(F(S) \setminus \partial_A(F(S))| \\
&\geq |F(S)| - |\partial_{A+D}(F(S))|,
\end{aligned}$$

which proves the lemma. $\square$

1.3.1. *Key lemma.* Roughly speaking, the following lemma states that if the intersection of $\mathcal{D}$ with two standard Følner sets in SOL of the same size is radically different, then the companion quasi-isometry to any biLipschitz map $f : \mathcal{D} \to \Gamma$ must map these Følner sets to boxes of sufficiently different sizes. This is the lemma that motivates the construction of $\mathcal{D}$ in the previous section.

Lemma 1.3. *There exist $M_0 > 0$ and $\epsilon_0 > 0$, both depending on K, C, satisfying the following: Let $M := N_0^{2^m} > M_0$ and suppose S^1, S^2 are translates of the standard Følner set S_M with $|\mathcal{D} \cap S^1| = d_1 |Q_0^m|$ and $|\mathcal{D} \cap S^2| = d_2 |Q_0^m|$. Suppose $f : \mathcal{D} \to \Gamma$ is K-biLipschitz, and that the companion quasi-isometry F satisfies*

$$F(S^1) \text{ is isometric to } U_{rM} \times U_s \times [0, \log M)$$

and

$$F(S^2) \text{ is isometric to } U_{r'M} \times U_s \times [0, \log M).$$

Then $|r - r'| > \epsilon_0$.

Proof. By Lemma 1.2 we have that

$$|F(S^i)| - |\partial_{A+D} F(S^i)| \leq |f(\mathcal{D} \cap S^i)| \leq |F(S^i)| + |\partial_{A+D} F(S^i)|$$

for $i = 1, 2$. We rewrite this as

$$|F(S^i)| - |\partial_{A+D} F(S^i)| \leq d_i |Q_0^m| \leq |F(S^i)| + |\partial_{A+D} F(S^i)|,$$

so that

$$\frac{|F(S^i)|}{|Q_0^m|} \left(1 - \frac{|\partial_{A+D} F(S^i)|}{|F(S^i)|}\right) \leq d_i \leq \frac{|F(S^i)|}{|Q_0^m|} \left(1 + \frac{|\partial_{A+D} F(S^i)|}{|F(S^i)|}\right). \tag{2}$$

Recall that $|F(S^1)| = rsM \log(M)$, $|F(S^2)| = r' s M \log(M)$ and $|Q_0^m| = M \log_{N_0}(M) \frac{|Q_0^0|}{N_0} = M \log_{N_0}(M) \frac{n}{N_0}$, hence

$$\frac{|F(S^1)|}{|Q_0^m|} = rs \frac{N_0}{n \log(N_0)} \quad \text{and} \quad \frac{|F(S^2)|}{|Q_0^m|} = r' s \frac{N_0}{n \log(N_0)}.$$

Now, since $F(S^1)$ and $F(S^2)$ are both Følner sets, we know that for any $\delta > 0$ there exists M_0 such that if $M > M_0$, then for both $i = 1$ and $i = 2$ we have

$$\frac{|\partial_{A+D} F(S^i)|}{|F(S^i)|} < \delta.$$

Notice that the constant M_0 depends on δ and the quasi-isometry constants K, C. We fix $\delta > 0$ so that $|d_1 - d_2| > \frac{3L^2\delta N_0}{n\log(N_0)}$. For $i = 1$, Equation (2) becomes

$$rs\frac{N_0}{n\log(N_0)}(1-\delta) \le d_1 \le rs\frac{N_0}{n\log(N_0)}(1+\delta),$$

and for $i = 2$,

$$r's\frac{N_0}{n\log(N_0)}(1-\delta) \le d_2 \le r's\frac{N_0}{n\log(N_0)}(1+\delta).$$

Thus, as $s \le L$ and $r \le L$,

$$\begin{aligned} d_2 - d_1 &\le r's\frac{N_0}{n\log(N_0)}(1+\delta) - rs\frac{N_0}{n\log(N_0)}(1-\delta) \\ &\le (r'-r)s\frac{N_0}{n\log(N_0)} + \delta s\frac{N_0}{n\log(N_0)}(r'+r) \\ &\le (r'-r)s\frac{N_0}{n\log(N_0)} + \delta 2L^2\frac{N_0}{n\log(N_0)}. \end{aligned}$$

Therefore, as $s \le L$, for $\varepsilon_0 = \frac{\delta L}{2}$ we must have $|r - r'| > \varepsilon_0$. □

1.4. **No K-bilipschitz maps.** We now show by contradiction that for any K there is no K-biLipschitz map $f : \mathcal{D} \to \Gamma$. Recall that each K-biLipschitz map f is bounded distance A from a companion map of the form $F(x,y,t) = (f_\ell(x), f_u(y), t)$, where f_ℓ, f_u are L-biLipschitz maps of $\mathbb{R}$.

Lemma 1.4. *Let $f : \mathcal{D} \to \Gamma$ be a map such that for all $i = 1, \dots, M$,*

$$\frac{|f_\ell((i-1)M) - f_\ell(iM)|}{M} \le (1+\lambda)\frac{|f_\ell(0) - f_\ell(M^2)|}{M^2}. \tag{3}$$

Then for some k we have both

$$\frac{|f_\ell((k-1)M) - f_\ell(kM)|}{M} \ge (1-\lambda)\frac{|f_\ell(0) - f_\ell(M^2)|}{M^2},$$

$$\frac{|f_\ell(kM) - f_\ell((k+1)M)|}{M} \ge (1-\lambda)\frac{|f_\ell(0) - f_\ell(M^2)|}{M^2}.$$

Proof. If at least half of the intervals get stretched by less than $(1-\lambda)$, we must have some other interval stretched by more than $(1+\lambda)$, which is impossible by our hypothesis. Therefore, by the pigeonhole principle, there are at least two consecutive intervals that get stretched by more than $(1-\lambda)$. □

Lemma 1.5. *There exist λ_0, M_0 such that if $M = N_0^{2^m} \ge M_0$, $\lambda \le \lambda_0$ and*

$$\frac{|f_\ell((i-1)M) - f_\ell(iM)|}{M} \le (1+\lambda)\frac{|f_\ell(0) - f_\ell(M^2)|}{M^2} \tag{4}$$

holds for $i = 1, \dots, M$, then $f : \mathcal{D} \to \Gamma$ cannot be K-biLipschitz.

Proof. The choice of M defines Følner sets for $i = 1, \ldots, M$:

$$S^i = [(i-1)M, iM) \times [0,1) \times [0, \log M).$$

Since $M = N_0^{2^m}$, all S^i are subsets of S_{M^2}.

Using Lemma 1.4, we find k such that both inequalities below hold:

$$\frac{|f_\ell((k-1)M) - f_\ell(kM)|}{M} \geq (1-\lambda)\frac{|f_\ell(0) - f_\ell(M^2)|}{M^2}, \tag{5}$$

$$\frac{|f_\ell(kM) - f_\ell((k+1)M)|}{M} \geq (1-\lambda)\frac{|f_\ell(0) - f_\ell(M^2)|}{M^2}. \tag{6}$$

Now either $|\mathcal{D} \cap S^k| = |Q_1^m| = d_1|Q_0^m|$ and $|\mathcal{D} \cap S^{k+1}| = |Q_2^m| = d_2|Q_0^m|$, or vice versa. At the same time, up to post-composition with isometries, we have that for some r, r',

$$F(S_k) = U_{rM} \times U_s \times [0, \log M)$$

and

$$F(S_{k+1}) = U_{r'M} \times U_s \times [0, \log M).$$

By Equations (4), (5) and (6),

$$|r - r'| \leq (1+\lambda)\frac{|f_\ell(0) - f_\ell(M^2)|}{M^2} - (1-\lambda)\frac{|f_\ell(0) - f_\ell(M^2)|}{M^2} \leq 2\lambda L.$$

Let M_0 and ϵ_0 be as specified by Lemma 1.3 and let $\lambda_0 = \frac{\epsilon_0}{2L}$. Then we have that if $M \geq M_0$ and $\lambda \leq \lambda_0$, then $f : \mathcal{D} \to \Gamma$ cannot be K-biLipschitz. □

Lemma 1.6. *Suppose $f_\ell : \mathbb{R} \to \mathbb{R}$ is a map. If there exist M_0 and λ_0 such that for all $M \geq M_0$ and $\lambda \leq \lambda_0$ we have that for at least one $i \in \{1, \ldots, M\}$,*

$$\frac{|f_\ell((i-1)M) - f_\ell(iM)|}{M} > (1+\lambda)\frac{|f_\ell(0) - f_\ell(M^2)|}{M^2},$$

then f_ℓ cannot be L-Lipschitz for any L.

Proof. Suppose f_ℓ is L-Lipschitz. Pick t such that

$$\frac{(1+\lambda)^t}{L} > L,$$

and recursively define $M_j = M_{j-1}^2$ for $j = 1, \ldots, t$. By assumption, for some i_t we must have

$$\frac{|f_\ell((i_t-1)M_{t-1}) - f_\ell(i_t M_{t-1})|}{M_{t-1}} > (1+\lambda)\frac{|f_\ell(0) - f_\ell(M_t)|}{M_t}.$$

Repeating this process, we get

$$\begin{aligned}
\frac{|f_\ell((i_1-1)M_0) - f_\ell(i_1 M_0)|}{M_0} &> (1+\lambda)\frac{|f_\ell((i_2-1)M_1) - f_\ell(i_2 M_1)|}{M_1} \\
&\vdots \qquad\qquad \vdots \\
&> (1+\lambda)^2\frac{|f_\ell((i_t-1)M_2) - f_\ell(i_t M_{t-1})|}{M_{t-1}} \\
&> (1+\lambda)^t\frac{|f_\ell(0) - f_\ell(M_t)|}{M_t}.
\end{aligned}$$

But then

$$L \geq \frac{|f_\ell((i_1-1)M_0) - f_\ell(i_1 M_0)|}{M_0} > (1+\lambda)^t \frac{|f_\ell(0) - f_\ell(M_t)|}{M_t} \geq \frac{(1+\lambda)^t}{L} > L,$$

which is a contradiction. □

Theorem 1.7. *Any lattice in SOL contains a non-rectifiable Delone subset.*

Proof. Let $\mathcal{D} \subset \Gamma$ be the set constructed in Section 1.1 and suppose that $f : \mathcal{D} \to \Gamma$ is a K-biLipschitz map. Let $F(x,y,t) = (f_\ell(x), f_u(y), t)$ be the companion map to f. Then f_ℓ must be L-biLipschitz for some L. Now by Lemma 1.5, we must have that there exist M_0 and λ_0 such that if $M \geq M_0$ and $\lambda \leq \lambda_0$, then for some $i \in \{1, \dots, M\}$,

$$\frac{|f_\ell((i-1)M) - f_\ell(iM)|}{M} > (1+\lambda)\frac{|f_\ell(0) - f_\ell(M^2)|}{M^2}.$$

However, Lemma 1.6 would then show that in this case f_ℓ cannot be L-biLipschitz for any L, which is a contradiction. □

2. Higher Rank analogues of SOL

In this section we generalize our arguments from lattices in SOL to lattices in a class of abelian-by-abelian solvable Lie groups. Namely, let $G_\phi = \mathbb{R}^{n+1} \rtimes_\phi \mathbb{R}^n$, where $\phi : \mathbb{R}^n \to SL_{n+1}(\mathbb{R})$ can be simultaneously diagonalized so that for each $\mathbf{t} \in \mathbb{R}^n$ the map $\phi(\mathbf{t}) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is multiplication by the exponential of

$$\begin{pmatrix} \alpha_1(\mathbf{t}) & 0 & \cdots & 0 \\ 0 & \alpha_2(\mathbf{t}) & 0 & \vdots \\ \vdots & 0 & \vdots & 0 \\ 0 & \cdots & 0 & \alpha_{n+1}(\mathbf{t}) \end{pmatrix}.$$

We call $\alpha_i : \mathbb{R}^n \to \mathbb{R}$ the *roots* associated to ϕ. We consider the case where α_i has the form $\alpha_i(\mathbf{t}) = a_i t_i$ for $i \leq n$, with $a_i > 0$, so that $\alpha_{n+1}(\mathbf{t}) = -(a_1 t_1 + \cdots + a_n t_n)$. Our theorem also holds in the slightly more general case where some of the a_i are negative but we omit this more general case primarily for the sake of ease of notation. In order to be able to construct tilings, however, we further restrict the a_i so there exists a $t \in \mathbb{R}$ such that $e^{a_i t} \in 2\mathbb{Z}$ for all i. We call such an abelian-by-abelian Lie group *even-scaling*.

In the case where, for each i, the *rootspace* V_{α_i} associated to α_i (i.e. the subspace on which ϕ acts by multiplication by $e^{\alpha_i(\mathbf{t})}$ for all $\mathbf{t} \in \mathbb{R}^n$) is one dimensional, such an abelian-by-abelian Lie group will be called *boundary one-dimensional*. We will write x_i for the coordinate representing V_{α_i}. From Peng [P1, P2], we have the following theorem on the structure of self quasi-isometries of such a G_ϕ.

Theorem 2.1 (Peng). *Any self (K,C) quasi-isometry of $G_\phi = \mathbb{R}^{n+1} \rtimes_\phi \mathbb{R}^n$ is, up to permuting the rootspaces, at bounded distance A from a map of the form*

$$(f_1 \times \cdots \times f_{n+1}) \times id$$

where f_i is a L-biLipschitz map of V_{α_i}.

Again, we have that for any Delone set $\mathcal{D} \subset G_\phi$ and any lattice $\Gamma \subset G_\phi$, every biLipschitz map $f : \mathcal{D} \to \Gamma$ induces a self quasi-isometry of G_ϕ that is bounded distance from a map of the form specified by Theorem 2.1. As before, we call F the *companion* quasi-isometry to f. Without loss of generality, we can assume that F does not permute the root spaces. Additionally, we can assume that f coarsely fixes the identity, and then $L = L(K, C)$ and $A = A(K, C)$.

2.1. **Følner sets.** We define standard Følner sets following (with slight modifications) the definitions given in 2.2.4 of [P1], where these Følner sets are called *boxes*. We first chose $\Omega = [0,1]^n \subset \mathbb{R}^n$ and note that Ω satisfies the *well rounded* condition of 2.2.4 of [P1]. Note also that for $i \leq n$ we have $\alpha_i(\Omega) = [0, a_i]$, whereas $\alpha_{n+1}(\Omega) = [-\sum_{i=1}^n a_i, 0]$. We define $a_{n+1} := 0$ to simplify notation below.

Let $b_{\alpha_j}(t) = [0,t] \subset V_{\alpha_j}$. (This slightly differs from the definition of $b_{\alpha_j}(t)$ in [P1]). Define $B(\Omega)$, the *box* associated to Ω, as the union of left translates of Ω over all elements of $\prod_{j=1}^{n+1} b_{\alpha_j}(e^{\max \alpha_j(\Omega)})$, that is,

$$B(\Omega) := \left(\prod_{j=1}^{n+1} b_{\alpha_j}(e^{\max \alpha_j(\Omega)}) \right) \Omega.$$

With a similar definition for $r\Omega$, we get that

$$B(r\Omega) = \left(\prod_{j=1}^{n+1} b_{\alpha_j}(e^{ra_j}) \right) r\Omega = \left(\prod_{j=1}^{n+1} [0, e^{ra_j}] \right) r\Omega = \left(\prod_{j=1}^{n} [0, e^{ra_j}] \right) \times [0,1] \times r\Omega.$$

Now, since the volume element is given by

$$e^{-\alpha_1(\mathbf{t})} dx_1 \wedge \cdots \wedge e^{-\alpha_{n+1}(\mathbf{t})} dx_{n+1} \wedge d\mathbf{t} = dx_1 \wedge \cdots \wedge dx_{n+1} \wedge d\mathbf{t},$$

the volume of a box $B(r\Omega)$ is given by

$$|B(r\Omega)| = \left(\prod_{j=1}^{n} e^{ra_j} \right) r^n |\Omega| = \left(\prod_{j=1}^{n} e^{ra_j} \right) r^n.$$

Lemma 2.2 (Lemma 2.2.7 in [P1])**.**

$$\left| \frac{\partial_\epsilon B(r\Omega)}{B(r\Omega)} \right| = O\left(\frac{\epsilon}{\operatorname{diam}(B(r\Omega))} \right). \tag{7}$$

Since a companion quasi-isometry stretches the various root spaces while keeping Ω fixed, we need to understand what happens to the volume and boundary of the box when we alter $B(r\Omega)$ slightly by keeping Ω fixed but changing the size of the intervals in $\mathbb{R}^{n+1}$.

Given $u := (u_1, \ldots, u_{n+1})$, define the *u-modified box* to be

$$B_u(r\Omega) := \left(\prod_{j=1}^{n+1} [0, e^{ra_j + u_j}] \right) r\Omega.$$

Then

$$|B_u(r\Omega)| = \left(\prod_{j=1}^{n+1} e^{ra_j + u_j} \right) r^n |\Omega| = e^{u_1 + \cdots + u_{n+1}} |B(r\Omega)|.$$

The following lemma shows that we also get behavior similar to Equation (7).

Lemma 2.3. *Let* $\|u\| := \sum |u_i|$. *Then*

$$\frac{|\partial_\epsilon B_u(r\Omega)|}{|B_u(r\Omega)|} = O\left(\frac{\epsilon e^{\|u\|}}{\mathrm{diam}(B(r\Omega))}\right). \tag{8}$$

Proof. We follow the same proof as in [P1] and bound the ratio of $\frac{|\partial_\epsilon B_u(r\Omega)|}{|B_u(r\Omega)|}$ in terms of $\frac{|\partial_\epsilon B(r\Omega)|}{|B(r\Omega)|}$. This will allow us to apply Lemma 2.2.7 of [P1].

First note that the boundary $|\partial_\epsilon B_u(r\Omega)|$ can be decomposed as follows:[1]

$$\left|\partial_\varepsilon\left(\prod_j [0, e^{ra_j+u_j}](r\Omega)\right)\right| = \underbrace{\left|\partial_\varepsilon\left(\prod_j [0, e^{ra_j+u_j}]\right)(r\Omega)\right|}_{(1)} + \underbrace{\left|\left(\prod_j [0, e^{ra_j+u_j}]\right)\partial_\varepsilon(r\Omega)\right|}_{(2)}.$$

When $u_i = 0$ for all i, we are in the case of Lemma 2.2.7 of [P1], and we will refer to these terms as $(1)'$ and $(2)'$. Next we estimate each term separately:

$$(2)\ :\ \left|\left(\prod_j [0, e^{ra_j+u_j}]\partial_\varepsilon(r\Omega)\right)\right| = \left(\prod_j e^{ra_j+u_j}\right) r^{n-1}|\partial_\varepsilon\Omega| = e^{u_1+\cdots u_{n+1}}\left|\left(\prod_j [0, e^{ra_j}]\partial_\varepsilon(r\Omega)\right)\right|$$

$$(1)\ :\ \left|\partial\left(\prod_j [0, e^{ra_j+u_j}]\right)(r\Omega)\right|$$

$$\begin{aligned}
&= 2\varepsilon\sum_{j=1}^{n+1}\int_{\mathbf{t}\in r\Omega}\underbrace{\int_0^{e^{ra_1+u_1}}\cdots\int_0^{e^{ra_{n+1}+u_{n+1}}} e^{-\alpha_1(\mathbf{t})}dx_1\dots e^{-\alpha_{n+1}(\mathbf{t})}dx_{n+1}}_{i\neq j}\,d\mathbf{t}\\
&= 2\varepsilon\sum_{j=1}^{n+1}\left(\prod_{i\neq j} e^{ra_i+u_i}\right)\int_{\mathbf{t}\in r\Omega}\underbrace{e^{-\alpha_1(\mathbf{t})}dx_1\dots e^{-\alpha_{n+1}(\mathbf{t})}dx_{n+1}}_{i\neq j}\,d\mathbf{t}\\
&= 2\varepsilon\sum_{j=1}^{n+1}\left(\prod_{i\neq j} e^{ra_i+u_i}\right)\int_{\mathbf{t}\in r\Omega} e^{\alpha_j(\mathbf{t})}d\mathbf{t}\\
&= 2\varepsilon\sum_{j=1}^{n}\left(\prod_{i\neq j} e^{ra_i+u_i}\right) r^{n-1}\int_0^r e^{a_jt_j}\,dt_j + 2\varepsilon\left(\prod_{i=1}^{n} e^{ra_i+u_i}\right)\int_{\mathbf{t}\in r\Omega} e^{-t_1a_1}e^{-t_2a_2}\cdots e^{-t_na_n}\,d\mathbf{t}\\
&= 2\varepsilon\sum_{j=1}^{n}\left(\prod_{i\neq j} e^{ra_i+u_i}\right) r^{n-1}a_j^{-1}(e^{ra_j}-1) + 2\varepsilon\left(\prod_{i=1}^{n} e^{ra_i+u_i}\right)(-1)^n a_1^{-1}\cdots a_n^{-1}(e^{-ra_1}-1)\cdots(e^{-ra_n}-1)\\
&= 2\varepsilon\sum_{j=1}^{n}\left(\prod_{i\neq j} e^{ra_i+u_i}\right) r^{n-1}a_j^{-1}(e^{ra_j}-1) + 2\varepsilon\left(\prod_{i=1}^{n} -a_i^{-1}(e^{-ra_i}-1)e^{ra_i+u_i}\right)\\
&= 2\varepsilon e^{u_1+\cdots+u_{n+1}} r^{n-1}\sum_{j=1}^{n} e^{-u_j}a_j^{-1}(e^{ra_j}-1)\left(\prod_{i\neq j} e^{ra_i}\right) + 2\varepsilon e^{u_1+\cdots+u_{n+1}}\left(e^{-u_{n+1}}\prod_{i=1}^{n} a_i^{-1}(e^{ra_i}-1)\right).
\end{aligned}$$

[1]Strictly speaking, there is an extra term arising from combination of points in $\partial_\varepsilon\left(\prod_j [0, e^{ra_j+u_j}]\right)$ and $\partial_\varepsilon(r\Omega)$, but the corresponding volume is negligeable.

Thus, in our case we have

$$\frac{|\partial_\epsilon B_u(r\Omega)|}{|B_u(r\Omega)|} = \frac{(1)}{|B_u(r\Omega)|} + \frac{(2)}{|B_u(r\Omega)|} = \frac{(1)}{e^{u_1+\cdots+u_{n+1}}|B(r\Omega)|} + \frac{(2)}{e^{u_1+\cdots+u_{n+1}}|B(r\Omega)|}.$$

In the second term, the expression $e^{u_1+\cdots u_{n+1}}$ cancel, thus giving

$$\frac{(2)}{|B_u(r\Omega)|} = \frac{(2)'}{|B(r\Omega)|}.$$

In the first term, if for all i we have $u_i \geq 0$, then

$$\frac{(1)}{|B_u(r\Omega)|} \leq \frac{(1)'}{|B(r\Omega)|}.$$

In the opposite case, if for all i we have $u_i < 0$, then

$$\frac{(1)}{|B_u(r\Omega)|} \leq e^{-(u_1+\cdots+u_{n+1})}\frac{(1)'}{|B(r\Omega)|}.$$

In general,

$$\frac{|\partial_\epsilon B_u(r\Omega)|}{|B_u(r\Omega)|} \leq e^{|u_1|+\cdots+|u_{n+1}|}\frac{|\partial_\epsilon B(r\Omega)|}{|B(r\Omega)|} = O\left(\frac{\epsilon e^{\|u\|}}{\operatorname{diam}(B(r\Omega))}\right),$$

as announced. □

The following corollary states that for $n = 2$ and $a_1 = 1, a_2 = -1$, this covers the case of SOL.

Corollary 2.4. *Let $G_\phi = \mathbb{R}^2 \rtimes \mathbb{R}$ be SOL. Then*

$$\frac{|\partial_\varepsilon B_u(r\Omega)|}{|B_u(r\Omega)|} \leq 2\frac{(e^{-u_1} + e^{-u_2})}{r} + \frac{1}{e^r}.$$

Proof. For SOL, a box $B(\Omega)$ is given by $\Omega = [0,1]$, so that $B(r\Omega) = [0,e^r]\times[0,1]\times[0,r]$, with volume re^r. Likewise, for $u := (u_1,u_2)$, we have

$$B_u(r\Omega) = [0, e^{r+u_1}] \times [0, e^{u_2}] \times [0, r],$$

with volume $|B_u(r\Omega)| = e^{u_1+u_2}re^r$. By investigating the calculations in our previous lemma, we see that

$$|\partial B_u(r\Omega)| = 2e^{u_1+u_2}(e^{-u_1}(e^r-1) + e^{-u_2}(e^r-1) + r)$$

So that

$$\begin{aligned} \frac{|\partial B_u(r\Omega)|}{|B_u(r\Omega)|} &= \frac{2e^{u_1+u_2}(e^{-u_1}(e^r-1) + e^{-u_2}(e^r-1) + r)}{e^{u_1+u_2}re^r} \\ &= \frac{2((e^{-u_1}+e^{-u_2})(e^r-1)+r)}{re^r} \\ &\leq 2\frac{(e^{-u_1}+e^{-u_2})}{r} + \frac{1}{e^r}, \end{aligned}$$

as announced. □

Notice that for $N = e^r$ (which fits with the set S_N previously considered in SOL), this ratio estimate becomes $2\frac{(e^{-u_1}+e^{-u_2})}{\log N} + \frac{1}{N} \leq 3\frac{(e^{-u_1}+e^{-u_2})}{\log N}$.

2.2. **Tiling.** Our tiling is an extension of the tiling of SOL. As mentioned above, for our tiling to work we need to chose a_i for which there exists $t \in \mathbb{R}$ such that $e^{a_i t} \in 2\mathbb{Z}$ for all i. If we define a standard Følner set by

$$S_t = \prod_{j=1}^{n+1} [0, e^{a_j t}] \times \prod_{i=1}^{n} [0, t],$$

then S_{2t} can be tiled by $2^n e^{t(\Sigma_{i=1}^n a_i)}$ many translates of S_t. (We have changed notation slightly from the case of SOL replacing $\log N_0$ by t since we must account for the different weights a_i.) Now we define our basic tiles Q_0, Q_1, Q_2, Q_3, Q_4 as follows. We start as before by choosing $t = T$ large enough so that $S_T \cap \Gamma$ contains at least 12 points that are at least distance C from the boundary of S_T. We pick 12 of these points and call the result Q_0. Then, for $j = 1, 2, 3, 4$, we chose $Q_j \subset Q_0$ to have density d_j, where d_j satisfies the conditions specified before in Section 1.1. (For example, $d_1 = 1/3, d_2 = 1/2, d_3 = 1/4, d_4 = 7/12$.)

Again, to define $Q_j^1 \subset S_{2T}$ for $j = 0, 1, 2, 3, 4$ from our basic tiles, we tile S_{2T} by translates of the Q_j's as follows. First we tile S_{2T} by translates of S_T. Replacing each translate of S_T with a translate of Q_0 (by the same element) defines the tile Q_0^1. For the other tiles we proceed as follows. Any translate that has as its first coordinate the interval $[ke^{a_1 T}, (k+1)e^{a_1 T}]$ with k even is colored red. If k is odd this translate is colored blue. The rest of the translates are colored green.

Claim. Half of the tiles are in S_{2T} are colored green, a quarter red and a quarter blue.

Proof. We have $S_{2T} = (\prod_{j=1}^{n+1} [0, e^{a_j 2T}]) \times [0, 2T]^n \subset \mathbb{R}^{n+1} \rtimes \mathbb{R}^n$ and the translates of S_T that cover S_{2T} correspond to sub-cubes of $[0, 2T]^n$, where each interval in the sub-cube is either $[0, T]$ or $[T, 2T]$. The translates of S_T with first coordinate $[ke^{a_1 T}, (k+1)e^{a_1 T}]$ correspond to those translates with sub-cube with first coordinate $[0, T]$, which is exactly half of all translates. Half of these have k odd and half have k even. $\square$

As before, to form Q_1^1 we replace the red translates with translates of Q_1, the blue translates with translates of Q_2, and the green translates with translates of Q_3. To form Q_2^1, we do the same except we replace the green translates with translates of Q_4. The sets Q_3^1, Q_4^1 are formed by replacing all of the colors with translates of Q_3 and Q_4 respectively. Then, as before, $|Q_j^1| = d_j |Q_0^1|$ for all $j = 1, 2, 3, 4$. Iterating this procedure allows us to define Q_j^m for any m. Finally, moving points a distance $\leq C$ apart if necessary, this defines for us a Delone set $\mathcal{D} \subset \Gamma$. (As before, whichever parts of G_ϕ we have not colored we can just replace with their intersection with Γ).

Next we need an analogue of Lemma 1.3. In the statement of this lemma, we define for any $t = 2^m T$ the set $\bar{S}_t := Q_0^m$. Then $|\bar{S}_t| = C' |S_t|$ for some constant C' depending on $|Q_0|$.

Lemma 2.5. *There exist $t_0 > 0$ and $\epsilon_0 > 0$, both depending on K, C, satisfying the following: Suppose that $t > t_0$ and that S^1, S^2 are translates of the standard Følner set S_t where $t = 2^m T$ for some m, and $|\mathcal{D} \cap S^i| = d_i |\bar{S}_t|$. Suppose also that $f : \mathcal{D} \to \Gamma$ is K-biLipschitz, and that the companion quasi-isometry F satisfies, for $i = 1, 2$ and up to postcomposition with isometries,*

$$F(S^i) = B_{u^i}(t\Omega).$$

Then $\|u^1 - u^2\| > \epsilon_0$.

Proof. Recall that $|S_t| = (\prod_{j=1}^{n+1} e^{a_j t})t^n$. By Lemma 1.2, we have that

$$|F(S^i)| - |\partial_{A+D} F(S^i)| \le |f(\mathcal{D} \cap S^i)| \le |F(S^i)| + |\partial_{A+D} F(S^i)|.$$

Hence,

$$|B_{u^i}(t\Omega)| - |\partial_{A+D} B_{u^i}(t\Omega)| \le C' d_i |S_t| \le |B_{u^i}(t\Omega)| + |\partial_{A+D} B_{u^i}(t\Omega)|,$$

and so

$$\frac{|B_{u^i}(t\Omega)|}{|S_t|} - \frac{|B_{u^i}(t\Omega)|}{|S_t|} \frac{|\partial_{A+D} B_{u^i}(t\Omega)|}{|B_{u^i}(t\Omega)|} \le C' d_i \le \frac{|B_{u^i}(t\Omega)|}{|S_t|} + \frac{|B_{u^i}(t\Omega)|}{|S_t|} \frac{|\partial_{A+D} B_{u^i}(t\Omega)|}{|B_{u^i}(t\Omega)|}$$

that is,

$$\frac{|B_{u^i}(t\Omega)|}{|S_t|} \left(1 - \frac{|\partial_{A+D} B_{u^i}(t\Omega)|}{|B_{u^i}(t\Omega)|}\right) \le C' d_i \le \frac{|B_{u^i}(t\Omega)|}{|S_t|} \left(1 + \frac{|\partial_{A+D} B_{u^i}(t\Omega)|}{|B_{u^i}(t\Omega)|}\right).$$

By Equation (8), for $t > t_0$ large enough we have that for both $i = 1, 2$,

$$\frac{|\partial_{A+D} B_{u^i}(t\Omega)|}{|B_{u^i}(t\Omega)|} < \delta.$$

Thus,

$$\frac{|B_{u^i}(t\Omega)|}{|S_t|} (1 - \delta) \le d_i \le \frac{|B_{u^i}(t\Omega)|}{|S_t|} (1 + \delta),$$

and therefore

$$\begin{aligned} C'|d_1 - d_2| &\le \frac{|B_{u^1}(t\Omega)|}{|S_t|}(1+\delta) - \frac{|B_{u^2}(t\Omega)|}{|S_t|}(1-\delta) \\ &\le \left| \frac{|B_{u^1}(t\Omega)| - |B_{u^2}(t\Omega)|}{|S_t|} \right| + \delta \left| \frac{|B_{u^1}(t\Omega)| + |B_{u^2}(t\Omega)|}{|S_t|} \right|. \end{aligned}$$

Recall also that

$$|B_{u^i}(t\Omega)| = \left(\prod_{j=1}^{n+1} e^{ta_j + u_j^i}\right) t^n = e^{\sum_{j=1}^{n+1} u_j^i} \left(\prod_{j=1}^{n+1} e^{ta_j}\right) t^n = e^{\sum_{j=1}^{n+1} u_j^i} |S_t|,$$

hence

$$\begin{aligned} \left| \frac{|B_{u^1}(t\Omega)| \pm |B_{u^2}(t\Omega)|}{|S_t|} \right| &\le \left| e^{\sum_{j=1}^{n+1} u_j^1} \pm e^{\sum_{j=1}^{n+1} u_j^2} \right| \\ &\le e^{\sum_{j=1}^{n+1} u_j^i} \left| e^{\sum_{j=1}^{n+1} (u_j^1 - u_j^2)} \pm 1 \right| \\ &\le e^{\sum_{j=1}^{n+1} u_j^i} \left| e^{\|u^1 - u^2\|} \pm 1 \right| \\ &\le L^{n+1} \left| e^{\epsilon} \pm 1 \right|, \end{aligned}$$

where $\varepsilon := \|u^1 - u^2\|$. The last step follows from the fact that all boundary maps f_j are L-biLipschitz, and so $e^{u_j^i} \le L$ for all j.

Summarizing, we have

$$C'|d_1 - d_2| \le L^{n+1} |e^{\epsilon} - 1| + \delta L^{n+1} |e^{\epsilon} + 1|.$$

In this inequality, d_1, d_2, C' and L are given, and δ can be made arbitrarily small by choosing $t > t_0$ for a large enough t_0. Fixing such a δ very small, we conclude that $\epsilon > \epsilon_0$ for some appropriately chosen t_0, ϵ_0. $\square$

2.3. **No K-bilipschitz maps.** The following is a modification of Lemma 1.5.

Lemma 2.6. *There exist $\lambda_0 > 0$ and $M_0 > 0$ such that if*

$$\frac{|f_1((i-1)M) - f_1(iM)|}{M} \leq (1+\lambda)\frac{|f_1(0) - f_1(M^2)|}{M^2} \tag{9}$$

holds for $i = 1, \ldots, M$, where $\lambda < \lambda_0$ and $M := e^{a_1 t}$ (with $t = T2^m$) satisfies $M \geq M_0$, then $f : \mathcal{D} \to \Gamma$ cannot be K-biLipschitz.

Proof. The choice of M defines Følner sets for $i = 1, \ldots, M$:

$$S^i = [(i-1)M, iM] \times \left(\prod_{j=2}^{n+1}[0, e^{ta_j}]\right) \times [0,t]^n.$$

These are all subsets of $S_{T2^{m+1}}$. As in Lemma 1.4, there exists k such that both inequalities below hold:

$$|f_1((k-1)M) - f_1(kM)| \quad \geq \quad (1-\lambda)\frac{|f_1(0) - f_1(M^2)|}{M}, \tag{10}$$

$$|f_1(kM) - f_1((k+1)M)| \quad \geq \quad (1-\lambda)\frac{|f_1(0) - f_1(M^2)|}{M}. \tag{11}$$

Now either $|\mathcal{D} \cap S^k| = |Q_1^m| = d_1|\bar{S}_t|$ and $|\mathcal{D} \cap S^{k+1}| = |Q_2^m| = d_2|\bar{S}_t|$, or vice versa. We also have that for some u^1, u^2 satisfying $u_j^1 = u_j^2$ for $j = 2, \ldots, n+1$, the set $F(S^k)$ is a translate of $B_{u^1}(t\Omega)$, and $F(S^{k+1})$ is a translate of $B_{u^2}(t\Omega)$. By equations (9), (10) and (11),

$$|e^{u_1^1} - e^{u^2 - 1}| \quad \leq \quad (1+\lambda)\frac{|f_1(0) - f_1(M^2)|}{M^2} - (1-\lambda)\frac{|f_1(0) - f_1(M^2)|}{M^2} \leq 2\lambda L.$$

Let t_0 and ϵ_0 be as in Lemma 2.5, set $M_0 = e^{t_0 a_1}$, and assume $e^{ta_1} = M \geq M_0$. Then we have

$$e^{\varepsilon_0} - 1 \leq |e^{u_1^1} - e^{u_1^2}| \leq 2\lambda L.$$

However, if $\lambda \leq \lambda_0$ for a very small λ_0, this yields a contradiction. $\square$

Theorem 2.7. *If $\Gamma \subset G_\phi = \mathbb{R}^{n+1} \rtimes_\phi \mathbb{R}^n$ is a lattice (where G_ϕ is a boundary one-dimensional even scaling abelian-by-abelian solvable Lie group as defined above), then there exist non-rectifiable Delone sets in Γ.*

Proof. Let $\mathcal{D}$ as indicated above and suppose that $f : \mathcal{D} \to \Gamma$ is a K-biLipschitz map. Let $F = (f_1, \ldots, f_{n+1}, id)$ be the companion quasi-isometry to f. Then f_1 must be L-biLipschitz for some L. Now by Lemma 2.6 we must have that there exist M_0 and λ_0 such that if $M \geq M_0$ and $\lambda \leq \lambda_0$, then for some $i \in \{1, \ldots, M\}$,

$$\frac{|f_1((i-1)M) - f_1(iM)|}{M} > (1+\lambda)\frac{|f_1(0) - f_1(M^2)|}{M^2}.$$

But then proceeding as in Lemma 1.6, we see that f_1 cannot be L-biLipschitz for any L, which is a contradiction. $\square$

3. Solvable Baumslag-Solitar groups

In this section, we show how the methods of the first two sections can also be used to produce non-rectifiable Delone sets in the solvable Baumslag-Solitar groups

$$BS(1,m) = \left\langle t, a \mid tat^{-1} = a^m \right\rangle .$$

The Baumslag-Solitar groups are not cocompact lattices in any real Lie group. However, they can be viewed as cocompact lattices in the locally compact isometry group of a fibered product X_m given by the following diagram:

$$\begin{tikzcd} X_m \arrow[r, "\rho_1"] \arrow[d, "\rho_2"'] \arrow[dr, "\bar{h}"] & T_{m+1} \arrow[d, "h"] \\ \mathbb{H}^2_m \arrow[r, "b"] & \mathbb{R} \end{tikzcd}$$

Here, $\mathbb{H}^2_m$ is a hyperbolic plane with curvature $-\frac{1}{\ln(m)}$ having $b : \mathbb{H}^2_m \to \mathbb{R}$ as a Busemann function, and T_{m+1} is an $(m+1)$-valent simplicial tree, with $h : T_{m+1} \to \mathbb{R}$ a height function defined by fixing an orientation on edges such that each vertex has one incoming edge and m outgoing edges. We also refer to $b : \mathbb{H}^2_m \to \mathbb{R}$ and the induced map $\bar{h} : X_m \to \mathbb{R}$ as height functions on $\mathbb{H}^2_m$ and X_m respectively. Topologically, the space X_m can be identified with $T_{m+1} \times \mathbb{R}$, but metrically it is isometric to a family of hyperbolic planes glued together along horoball complements at integer heights. For a more detailed description of X_m see [FM1, FM2]. By fixing a base point x_0 with $\bar{h}(x_0) = 0$ and considering the orbit of x_0 under $BS(1,m)$, we can embed the Cayley graph of $BS(1,m)$ into X_m.

Remark. The group SOL has a similar description to the above with $\mathbb{H}^2_m$ and T_{m+1} replaced by two hyperbolic planes $\mathbb{H}^2$ and h replaced by $-b$, the negative of a Busemann function $b : \mathbb{H}^2 \to \mathbb{R}$.

Lemma 3.1. *There exists a metric on*

$$\{(x,y,t) \in \mathbb{R} \times \mathbb{Q}_m \times \mathbb{R}\}$$

and two (K,C) quasi-isometries with $K = C = 1$ that are coarse inverses of each other

$$\pi : \mathbb{R} \times \mathbb{Q}_m \times \mathbb{R} \to X_m, \quad \bar{\pi} : X_m \to \mathbb{R} \times \mathbb{Q}_m \times \mathbb{R},$$

such that $\bar{\pi}$ is injective on $BS(1,m) \subset X_m$.

The above lemma will be proved as part of the proof of the following theorem.

Theorem 3.2 (Farb-Mosher [FM2])**.** *For any (K,C) quasi-isometry $\phi : X_m \to X_m$, the map*

$$\pi \circ \phi \circ \bar{\pi} : \mathbb{R} \times \mathbb{Q}_m \times \mathbb{R} \to \mathbb{R} \times \mathbb{Q}_m \times \mathbb{R}$$

is at bounded distance $A = A(K,C)$ from a map F of the form

$$F(x,y,t) = (f_\ell(x), f_u(y), t),$$

where the f_ℓ, f_u are L-biLipschitz maps of $\mathbb{R}$ and $\mathbb{Q}_m$ respectively and where the constant L depends only on K and C provided that some base point is C-coarsely fixed by f.

Proof. All of this can be found in [FM1, FM2] albeit using slightly different language and so we reconcile these differences here. In [FM1], Farb and Mosher first define a *lower boundary* $\partial_\ell X_m$ and *upper boundary* $\partial_u X_m$ as equivalence classes of *vertical geodesics* in X_m. Vertical geodesics are those geodesics $\gamma(t)$ that project isometrically to $\mathbb{R}$ under $\bar{h}$ and are parametrized so that $\bar{h}(\gamma(t)) = t$. Two vertical geodesics are equivalent in the lower boundary if they stay a bounded distance apart in X_m as $t \to -\infty$, and they are equivalent in the upper boundary if they stay bounded distance apart as $t \to \infty$. One can identify $\partial_\ell X_m \simeq \partial\mathbb{H}^2_m \smallsetminus \{\infty\} \simeq \mathbb{R}$ and $\partial_u X_m \simeq \partial T_{m+1} \smallsetminus \{\infty\} \simeq \mathbb{Q}_m$. To specify a vertical geodesic in X_m, is it enough to chose $x \in \partial_\ell X_m \simeq \mathbb{R}$ and $y \in \partial_u X_m \simeq \mathbb{Q}_m$. To specify a point on that geodesic, one simply specifies its height $t \in \mathbb{R}$. This defines a natural projection map

$$\pi : \mathbb{R} \times \mathbb{Q}_m \times \mathbb{R} \to X_m,$$

where (x, y, t) is mapped to the the point at height t in X_m on the vertical geodesic with lower boundary endpoint $x \in \partial_\ell X_m$ and upper boundary endpoint $y \in \partial_u X_m$. Note that this map is not injective. Indeed any point (x, y', t) with $y' \in B_{\mathbb{Q}_m}(y, r)$ where $-\log_m r = t$ also defines the same point in X_m. We can define a nice coarse inverse map $\bar{\pi} : X_m \to \mathbb{R} \times \mathbb{Q}_m \times \mathbb{R}$ by sending $v \in X_m$ to an arbitrary point (x, y, t) with $\pi(x, y, t) = v$. The only choice one has in defining $\bar{\pi}$ is in the y coordinate. The following picture illustrates a possible choice for the $\bar{\pi}$ map restricted to the tree coordinate.

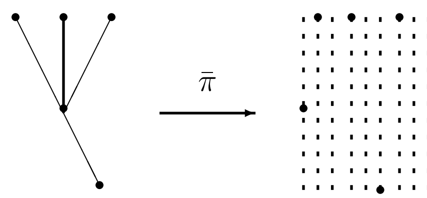

FIGURE 1. A possible choice for $\bar{\pi}$ shown only in projection onto T_{3+1} and $\mathbb{Q}_3 \times \mathbb{R}$.

We endow $\bar{X}_m$ with a metric that is coarsely equivalent to the induced metric from X_m:

$$d_{\bar{X}_m}\big((x_1, y_1, t_1), (x_2, y_2, t_2)\big) \coloneqq m^{-\frac{t_1+t_2}{2}}|x_1 - x_2| + m^{\frac{t_1+t_2}{2}} d_{\mathbb{Q}_m}(y_1, y_2) + |t_1 - t_2|.$$

In [FM1, FM2] it is shown that every quasi-isometry $\phi : X_m \to X_m$ preserves height level sets up to bounded distance and maps vertical geodesics to within bounded distance of vertical geodesics. They use this information to define induced boundary maps f_ℓ, f_u and conclude that these maps are biLipschitz. $\square$

We then set $\bar{X}_m \coloneqq \mathbb{R} \times \mathbb{Q}_m \times \mathbb{R}$ and identify $BS(1, m)$ with its image $\bar{\pi}(BS(1, m))$. If we choose our coordinates so that the identity in $BS(1, m)$ has coordinates $(0, 0, 0)$ then any $v \in BS(1, m)$ has coordinates (x, y, t) with $x \in \mathbb{Z}[\frac{1}{m}] \subset \mathbb{R}$, $y \in \mathbb{Z}[\frac{1}{m}] \subset \mathbb{Q}_m$ and $t \in \mathbb{Z}$.

Note also that there is an action of $BS(1, m)$ on $\bar{X}_m$ given by its action on vertical geodesics in X_m. Similarly to $\Gamma \subset SOL$, one can show (see Lemma 3.3 below) that Følner sets in $BS(1, m)$ can be given by intersecting with $BS(1, m)$ translates of the set

$$S_N \coloneqq [0, N) \times B(0, 1) \times [0, \log_m N),$$

where $B(0,1)$ is a ball of radius one in $\mathbb{Q}_m$. When N is a power of m we have

$$|BS(1,m) \cap S_N| = N \log_m N,$$

but otherwise $|BS(1,m) \cap S_N| = \lfloor N \rfloor \lfloor \log_m N \rfloor$. We will write $|S_N|$ for $|BS(1,m) \cap S_N|$ in either case.

The figure below shows a Følner set in $BS(1,2) \subset X_2$, where a portion of a $\mathbb{H}_2^2 \subset X_2$ component is on the left, a portion of the projected view onto T_{2+1} is on the right, and the projected view onto $\mathbb{H}_2^2$ (showing all points) is in the middle.

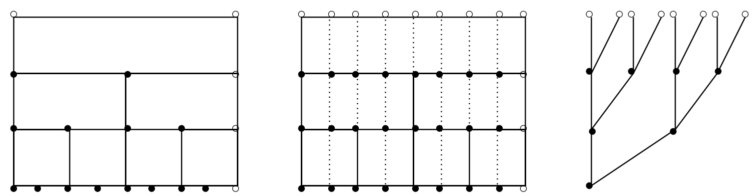

FIGURE 2. A Følner set in $BS(1,2)$ containing $3 \cdot 2^3$ elements.

The following lemma shows that $S_N \cap BS(1,m)$ is indeed a Følner sequence. The slightly more general statement will be useful in later proofs.

Lemma 3.3. *Suppose U_{rN} is an interval of length rN. Then any set of the form*

$$S'_N = U_{rN} \times B(0,1) \times [0, \log_m N)$$

for a fixed r is a Følner set.

Proof. To compute the boundary of S'_N it is useful to consider $\pi(S'_N)$. Doing this, and arguing as in Lemma 2.3, it becomes clear that the 1-boundary of S'_N can be decomposed into (1) and (2) below:

(1) $|\partial_1(U_{rN} \times B(0,1))|$: this is twice the number of vertices in a rooted tree of height $\log_m N - 1$ with branching constant m, that is, $2\frac{N-1}{m-1}$;
(2) $|\partial_1[0, \log_m N)|$: this is twice the number $\lfloor rN \rfloor$.

Since $BS(1,m)$ is generated by two elements, an obvious argument gives that the R-boundary of S'_N is bounded by

$$|\partial_R S'_N| \le e^{4R}\left(2\lfloor rN \rfloor + 2\frac{N-1}{m-1}\right).$$

Since the volume of S'_N is given by $\lfloor rN \rfloor \log_m N$, this yields the ratio

$$\frac{|\partial_R S'_N|}{|S'_N|} \le \frac{e^R(2\lfloor rN \rfloor + 2\frac{N-1}{m-1})}{\lfloor rN \rfloor \log_m N} \sim \frac{1}{\log_m N}$$

which goes to 0 as $N \to \infty$. □

Next we need to estimate the size of $|F(S_N) \cap BS(1,m)|$.

Lemma 3.4. *If $F(S_N) = U_{rN} \times U_s \times [0, \log_m N)$, then for large enough N,*

$$\left(\frac{1}{L^2} - \epsilon\right)|S_N| \le |F(S_N)| \le L^4 |S_N|,$$

where $\epsilon \to 0$ as $N \to \infty$.

Proof. Recall that $F = (f_\ell, f_u, id)$, where f_ℓ, f_u are L-biLipschitz. In particular, $1/L \le r \le L$. As noted in the proof of Proposition 4.3 in [D], the image of any ball in $\mathbb{Q}_m$ under an L-biLipschitz map is the disjoint union of finitely many balls of bounded size, where both the number of balls and their size are bounded by constants that depend on L and the size of the original ball. Specifically, $U_s = f_u(B(0,1)) = \bigsqcup_{i \in I} A_i$, where A_i are balls of size at least $|B(0,1)|/L = 1/L$. Indeed, if $z \notin B(0,1)$, then $d_{\mathbb{Q}_m}(z, B(0,1)) > 1$ and therefore, for some A_i,

$$d_{\mathbb{Q}_m}(f_u(z), A_i) = d_{\mathbb{Q}_m}(f_u(z), f_u(B(0,1))) > \frac{1}{L}. \tag{12}$$

This implies that $|A_i| > \frac{1}{L}$, otherwise there would be some y with $d_{\mathbb{Q}_m}(y, A_i) \le \frac{1}{L}$, and letting z be such that $y := f_u(z)$ (remind that f_u is surjective), this would contradict inequality (12).

Without loss of generality, the A_i can be chosen to be all of the same size. Then since $|I||A_i| = |f_u(B(0,1))|$, we must have $\frac{1}{L} \le |I||A_i| \le L$, and combining this with the inequality $|I|\frac{1}{L} < |I||A_i|$, we get that $|I| < L^2$. We can further assume that $|A_i| = m^{-j} \le 1$ where j depends only on L. (Since $|A_i| \le L$ we can always achieve this by writing each A_i as a union of at most L balls of size at most 1.) This increases $|I|$ by at most a factor of L, hence $|I| \le L^3$. There is also a lower bound $\frac{m^j}{L} \le |I|$, since $\frac{1}{L} \le |I||A_i|$.

Next we estimate the size of $U_{rN} \times A_i \times [0, \log_m N)$. First we estimate the size of the subset

$$U_{rN} \times A_i \times [-\log_m |A_i|, \log_m N) = U_{rN} \times A_i \times [j, \log_m N).$$

For this subset, the choice of $\bar{\pi}$ does not change the number of points in

$$BS(1,m) \cap (U_{rN} \times A_i \times [j, \log_m N),$$

which is exactly $\lfloor rNm^{-j} \rfloor (\log_m N - j)$. Therefore,

$$(rNm^{-j} - 1)(\log_m N - j) \le |U_{rN} \times A_i \times [j, \log_m N)| \le rNm^{-j}(\log_m N - j).$$

This gives us the lower bound

$$\begin{aligned}
|F(S_N)| &\ge \sum_{i \in I} (rNm^{-j} - 1)(\log_m N - j) \\
&\ge \frac{m^j}{L}(rNm^{-j} - 1)(\log_m N - j) \\
&\ge \left(\frac{r}{L} - \frac{m^j}{LN} - \frac{jr}{L \log_m N}\right)|S_N| \\
&\ge \left(\frac{1}{L^2} - \epsilon\right)|S_N|,
\end{aligned}$$

where the last inequality holds for large enough N for a prescribed $\varepsilon > 0$.

For the upper bound, we have to consider the extra points that may appear in the interval $[0, j)$. Here, for each A_i there will be at most $j(rN)$ points, so that

$$\begin{aligned}
|F(S_N)| &\leq \sum_{i \in I} \left(rNm^{-j}(\log_m N - j) + jrN \right) \\
&\leq L^3 rN \log_m N - L^3 rNj + L^3 jrN \\
&\leq L^3 r |S_N| \\
&\leq L^4 |S_N|
\end{aligned}$$

as desired. $\square$

Lemma 3.5. *The image of the standard Følner set S_N under F is also a Følner set.*

Proof. We use some of the work from the proof of the previous lemma. As before, let

$$F(S_N) = U_{rN} \times \bigsqcup_{i \in I} A_i \times [0, \log_m N),$$

with $|A_i| = m^{-j}$ and $\frac{m^j}{L} \leq |I| \leq L^3$. Fix i and let $S' = U_{rN} \times A_i \times [0, \log_m N)$. We will give an upper bound for $|\partial_R S'|$ and multiply this by $|I|$ to get an upper bound for $\partial_R F(S_N)$; then, we will combine it with the lower bound for $|F(S_N)|$ from Lemma 3.4 to get the desired result. To that end, we decompose S' as follows

$$S' = U_{rN} \times A_i \times [j, \log_m N) \bigcup U_{rN} \times A_i \times [0, j).$$

Note that $U_{rN} \times A_i \times [j, \log_m N)$ is a translate of the set

$$U_{rNm^{-j}} \times B(0,1) \times [0, \log_m N - j) = U_{rNm^{-j}} \times B(0,1) \times [0, \log_m (Nm^{-j})).$$

This allows us to appeal to Lemma 3.3 to get that

$$|\partial_R \left(U_{rN} \times A_i \times [j, \log_m N) \right)| \leq e^{4R} \left(2\lfloor rm^{-j} N \rfloor + 2\frac{Nm^{-j} - 1}{m - 1} \right).$$

Next, for $\partial_R(U_{rN} \times A_i \times [0, j))$ we have the estimate

$$|\partial_R(U_{rN} \times A_i \times [0, j))| \leq e^{4R} rNj,$$

since in particular $|U_{rN} \times A_i \times [0, j)| \leq rNj$. Putting everything together we get that

$$\frac{|\partial_R F(S_N)|}{|F(S_N)|} \leq \frac{e^{4R} \left(2\lfloor rm^{-j} N \rfloor + 2\frac{Nm^{-j}-1}{m-1} \right) + e^{4R} rNj}{\left(\frac{1}{L^2} - \varepsilon \right) |S_N|}$$

which goes to 0 as $N \to \infty$. $\square$

3.1. **Tiling.** Constructing $\mathcal{D}$ can be done exactly as for SOL. For N a power of m, we consider the set S_N, so that $S_N \cap BS(1, m)$ contains exactly $N \log_m N$ elements. As before, we can tile the set S_{N^2} by $2N$ translates of S_N in two layers of N copies each:

$$T_{k,1} \coloneqq g_{k,1} S_N = [(k-1)N, kN) \times B(0,1) \times [0, \log_m N),$$

$$T_{k,2} \coloneqq g_{k,2} S_N = [0, N^2) \times B\big(z_k, \frac{1}{N}\big) \times [\log_m N, 2\log_m N).$$

Here, both $g_{k,1}$ and $g_{k,2}$ lie in $BS(1, m)$, and the points $z_k = \sum a_i m^i \in \mathbb{Q}_m$ satisfy $a_i = 0$ for $i < 0$ and $i \geq \log_m N$, and $a_0, a_2, \ldots, a_{\log_m N - 1}$ lie in the set $\{0, \ldots, m-1\}$. (Note that there are $m^{\log_m N} = N$ possibilities for $(a_0, a_2, \ldots, a_{\log_m N - 1})$, hence N points z_k.)

So now we fix N_0 a power of m so that $|S_{N_0} \cap BS(1,m)| \geq 12$ and let $Q_0 = S_{N_0} \cap BS(1,m)$. (For $BS(1,2)$, the value of N_0 can be 8, for example.) Our four basic tiles Q_1, Q_2, Q_3, Q_4 can be defined by choosing $d_1 = 1/3, d_2 = 1/2, d_3 = 1/4, d_4 = 7/12$ of the points from Q_0. If m is even, then our tilling works exactly as before: to define Q_1^1 (resp. Q_2^1) we use $N_0/2$ translates of Q_1 (for each $T_{k,1}$ with k even), $N_0/2$ translates of Q_2 (for each k odd) and N_0 translates of Q_3 (resp. Q_4) for each $T_{k,2}$. However, when m is odd, N_0 will be odd as well, and so if we have $\lceil N_0/2 \rceil$ translates of Q_1 and $\lfloor N_0/2 \rfloor$ translates of Q_2, then if we replace all $T_{k,2}$ with Q_3 (resp. Q_4), the volume $|Q_1^1|$ (resp. $|Q_2^1|$) will not be exactly $d_1|Q_0^1|$ (resp. $d_2|Q_0^1|$). To solve this problem, by replacing one of the Q_3 (resp. Q_4) tiles with a tile of slightly different density, we can again ensure that $Q_j^1 = d_j|Q_0^1|$ for $j = 1, 2$. Specifically, we replace a Q_3 (resp. Q_4) tile with a tile of density $d_3 + \frac{1}{2}(d_2 - d_1)$ (resp. $d_4 + \frac{1}{2}(d_2 - d_1)$). One can check that if the d_j are given as above, then this new tile has density $1/3$ (resp. $2/3$). In both cases, Q_3^1 and Q_4^1 are defined as before, by replacing all the tiles with Q_3 (resp. Q_4).

Remark. When m is odd, there is another modification that needs to be made in the proof. Namely, Lemma 1.4 which we appeal to later in the proof does not hold with an odd number of intervals. However, it does hold if we modify the lower bounds to $(1 - 2\lambda)$. This will only change the constants slightly in the subsequent lemmas and propositions.

The following lemma, which is an analogue of Lemma 1.3, follows almost exactly as before.

Lemma 3.6. *There exist $M_0 > 0$ and $\epsilon_0 > 0$, both depending on K, C, satisfying the following: Suppose $M = m^j > M_0$ and suppose S^1, S^2 are standard Følner sets in $BS(1,m)$ that are translates of S_M with $|\mathcal{D} \cap S_1| = d_1|S_M|$ and $|\mathcal{D} \cap S_2| = d_2|S_M|$. Suppose also that $f : \mathcal{D} \to BS(1,m)$ is K-biLipschitz, and that its companion quasi-isometry F satisfies*

$$F(S^1) = U_{rM} \times U_s \times [0, \log_m M)$$

$$F(S^2) = U_{r'M} \times U_s \times [0, \log_m M)$$

where $U_s = \bigsqcup_{i \in I} A_i$. Then $|r' - r| > \epsilon_0$.

Proof. As before, since

$$F(S^i) \smallsetminus \partial_A F(S^i) \subseteq f(S^i \cap \mathcal{D}) \subseteq F(S^i) \cup \partial_A F(S^i),$$

we have that

$$|F(S^i)| - |\partial_A F(S^i)| \leq d_i |S_M| \leq |F(S^i)| + |\partial_A F(S^i)|,$$

and so

$$\frac{|F(S^i)|}{|S_M|}\left(1 - \frac{|\partial_A F(S^i)|}{|F(S^i)|}\right) \leq d_i \leq \frac{|F(S^i)|}{|S_M|}\left(1 + \frac{|\partial_A F(S^i)|}{|F(S^i)|}\right).$$

By Lemma 3.5, we have that for any $\delta > 0$ we can chose M large enough so that

$$\frac{|\partial_R F(S_M)|}{|F(S_M)|} < \delta \tag{13}$$

and by Lemma 3.4 we know that for $i = 1, 2$,

$$|F(S^i)| < L^4 |S_M|.$$

Thus,

$$\begin{aligned} d_1 - d_2 &\leq \frac{|F(S^1)|}{|S_M|}(1+\delta) - \frac{|F(S^2)|}{|S_M|}(1-\delta) \\ &\leq \frac{|F(S^1)| - |F(S^2)|}{|S_M|} + \delta\frac{|F(S^1)| + |F(S^2)|}{|S_M|} \\ &\leq \frac{rsM\log_m M - r'sM\log_m M}{M\log_m M} + 2\delta L^4. \\ &\leq (r-r')L + 2\delta L^4. \end{aligned}$$

Now pick $\delta < |d_1 - d_2|/3L^4$ and M_0 such that for $M > M_0$ Equation (13) holds. Then for $\epsilon_0 = \delta L^3$ we must have $|r - r'| > \epsilon_0$, otherwise the above inequality would be violated. □

3.2. **No K-bilipschitz maps.** We also need a slight modification of Lemma 1.5.

Lemma 3.7. *There exist $\lambda_0 > 0$ and $M_0 > 0$ such that if $\lambda < \lambda_0$, $m^j = M \geq M_0$ and*

$$\frac{|f_\ell((i-1)M) - f_\ell(iM)|}{M} \leq (1+\lambda)\frac{|f_\ell(0) - f_\ell(M^2)|}{M^2} \tag{14}$$

holds for $i = 1, \ldots, M$, then $f : \mathcal{D} \to BS(1,m)$ cannot be K-biLipschitz.

Proof. Again, the choice of M defines Følner sets for $i = 1, \ldots, M$:

$$S^i = [(i-1)M, iM) \times B(0,1) \times [0, \log_m M).$$

These are all subsets of S_{M^2}. By Lemma 1.4, we can find k such that both inequalities below hold:

$$|f_1((k-1)M) - f_1(kM)| \geq (1-\lambda)\frac{|f_1(0) - f_1(M^2)|}{M}, \tag{15}$$

$$|f_1(kM) - f_1((k+1)M)| \geq (1-\lambda)\frac{|f_1(0) - f_1(M^2)|}{M}. \tag{16}$$

Now either $|\mathcal{D} \cap S^k| = |Q_1^j| = d_1|S_{m^{j_0}}|$ and $|\mathcal{D} \cap S^{k+1}| = |Q_2^j| = d_2|S_{m^{j_0}}|$, or vice versa. By equations (14), (15) and (16),

$$|r - r'| \leq (1+\lambda)\frac{|f_1(0) - f_1(M^2)|}{M^2} - (1-\lambda)\frac{|f_1(0) - f_1(M^2)|}{M^2} \leq 2\lambda L.$$

Let M_0 and ϵ_0 be as in Lemma 3.6 and $\lambda_0 = \frac{\epsilon_0}{2L}$. Then we have that if $M \geq M_0$ and $\lambda \leq \lambda_0$ then $f : \mathcal{D} \to \Gamma$ cannot be K-biLipschitz. □

Theorem 3.8. *For any m, there exist non-rectifiable Delone sets in $BS(1,m)$.*

Proof. Let $\mathcal{D}$ as indicated above and suppose that $f : \mathcal{D} \to BS(1,m)$ is a K-biLipschitz map. Let $F = (f_\ell, f_u, Id)$ be the companion quasi-isometry to f. Then f_ℓ must be L-biLipschitz for some L. Now by Lemma 3.7, there exist M_0 and λ_0 such that if $M \geq M_0$ and $\lambda \leq \lambda_0$, then for some $i \in \{1, \ldots, M\}$,

$$\frac{|f_1((i-1)M) - f_1(iM)|}{M} > (1+\lambda)\frac{|f_1(0) - f_1(M^2)|}{M^2}.$$

But then Lemma 1.6 shows that f_ℓ cannot be L-biLipschitz for any L. □

Remark. All of the groups Γ we were able to treat in this paper had the following main feature in common: We were able to identify their quasi-isometry as products of bilipschitz maps on various *boundary* metric spaces with at least one of those spaces being $\mathbb{R}$. Using this information we were able to show that under a quasi-isometry Følner sets were mapped to Følner sets and then to construct a tiling of the space in a way that made it impossible for a bilipschitz map $f : \mathcal{D} \to \Gamma$ to have a boundary map that was biLipschitz in $\mathbb{R}$. It would be interesting to find examples of non-rectifiable Delone sets in abelian-by-abelian solvable Lie groups whose boundaries were not all one dimensional.

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