Quasisymmetric maps of boundaries of
amenable hyperbolic groups

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Abstract

In this paper we show that if $Y = N \times \mathbb{Q}_m$ is a metric space where $N$ is a connected, simply connected, nilpotent Lie group endowed with an \textit{admissible} metric then any quasisymmetric map of $Y$ is actually bilipschitz. A metric on $N$ is admissible if it makes $Y$ into a \textit{parabolic visual boundary} of a \textit{mixed type} locally compact amenable hyperbolic group. We also prove some rigidity results on uniform subgroups of bilipschitz maps of $Y$ in the case where $N = \mathbb{R}^n$.

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1 Introduction

Quasisymmetric maps were introduced in [BA56] as natural replacements for quasiconformal maps for metric spaces where classical quasiconformal maps do not make sense (see Section 2 for the definition). For many standard metric spaces (such as $\mathbb{R}^n$) quasisymmetric maps are much more abundant than bilipschitz maps. For $Y = N \times \mathbb{Q}_m$ this is not the case.

\textbf{Theorem 1} Let $N$ be a nilpotent Lie group with an admissible metric $d$ and $\mathbb{Q}_m$ the $m$-adics with the standard metric. Then any quasisymmetric map of $Y = N \times \mathbb{Q}_m$ onto itself is bilipschitz.

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Quasisymmetric maps are especially interesting since they are strongly linked to negative curvature geometry in that quasi-isometries of negatively curved spaces induce quasisymmetric maps of their visual boundaries. We are able to study $Y$ precisely because it is the parabolic visual boundary of a geodesic negatively curved space $X_{N,\varphi,m}$ that fibers over a simplicial $m+1$ valent tree $T_{m+1}$. Quasisymmetric maps of $Y = N \times \mathbb{Q}_m = \partial X_{N,\varphi,m}$ are induced by quasi-isometries of $X_{N,\varphi,m}$ while bilipschitz maps of $Y$ are induced by height-respecting quasi-isometries (see Section 3 for the definition). Using coarse topology provided by [FM00] we prove Theorem 1 by showing that all quasi-isometries of $X_{N,\varphi,m}$ are height-respecting.

This result should be compared to results of Xie in [SX, Xiec, Xiea, Xieb] which use the same dictionary but in the reverse direction to show that all quasi-isometries of certain negatively curved homogeneous spaces are height-respecting by proving directly that all quasisymmetric maps of their boundaries are bilipschitz.

In fact, Theorem 1 can be thought of as an extension of Xie’s results. In [Hei74], Heintze showed that all negatively curved homogeneous spaces can be given as solvable Lie groups (see Section 3.1 for details). In a similar spirit [CdCMT], Caprace et al. classify all locally compact amenable hyperbolic groups. They show that there are three types of non-elementary amenable hyperbolic locally compact groups: negatively curved homogeneous spaces, stabilizers of an end in the full automorphism group of a semi-regular locally finite tree, and combinations of the two via a warped product construction. It is this last type that we call mixed type. All mixed type amenable hyperbolic groups act properly on some $X_{N,\varphi,m}$.

**Remark.** Theorem 1 should hold in more generality. For example it should hold for any space whose hyperbolic cone (see [BS00]) satisfies the conditions imposed in Theorems 7.3 and 7.7 in [FM00]. (See Section 4 for the statements of these theorems.)

### 1.1 Other results.

Using the same arguments as Corollary 1.3 in [SX] we can also show the following:

**Corollary 2** There are no finitely generated groups quasi-isometric to $X_{N,\varphi,m}$. 

2
Furthermore, from the appendix of [FM98] we know that $Q_m$ and $Q_{m'}$ are bilipschitz equivalent only if $m = r^i$ and $m' = r^j$ for some common base $r$. This gives us a partial quasi-isometry classification result.

**Corollary 3** $X_{N,\varphi,m}$ is not quasi-isometric to $X_{N',\varphi',m'}$ if $m, m'$ are not powers of a common base.

A more detailed classification result can be deduced in many cases from [FM00, Pen11, Ahl02] but we will not give the details here except in case of $N = \mathbb{R}^n$. Note that if we specialize to $N = \mathbb{R}^n$ then $\varphi = \varphi_M$ is given by multiplication by an $n \times n$ matrix $M$ whose eigenvalues all have norm greater than one. Using work of [FM00] we get a full quasi-isometry classification in this case.

**Corollary 4** $X_{\mathbb{R}^n,\varphi_M,m}$ is quasi-isometric to $X_{\mathbb{R}^n,\varphi_{M'},m'}$ if and only if $m = r^i$, $m' = r^j$ and $M$ and $M'$ have absolute Jordan forms that are powers of each other.

We also extend results from [Dym10] and prove that certain groups of bilipschitz maps of $\mathbb{R}^n \times Q_m$ can be conjugated to be of a particularly nice form. We state the theorem here but all definitions are given in the appendix.

**Theorem 5** Let $U$ be a uniform separable subgroup of $\text{Bilip}_M(\mathbb{R}^n \times Q_m)$ that acts cocompactly on the space of distinct pairs of points of $\mathbb{R}^n \times Q_m$. Then $U$ can be conjugated into $\text{ASim}_M(\mathbb{R}^n \times Q_p)$ for some $p$.

The proof follows Theorem 2 in [Dym10] very closely which is why we relegate this result to the appendix and provide only a brief outline.

### 1.2 Outline

Following some preliminaries in Section 2 we study the geometry and boundaries of $X_{N,\varphi,m}$ in Section 3. We prove Theorem 1 in Section 4. In the appendix we prove Theorem 5.

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2 Preliminaries

Definition 1 A map \( f : X \to Y \) between metric spaces is called a

Quasisymmetric embedding if for some homeomorphism \( \eta : [0, \infty) \to [0, \infty) \)

\[
\frac{d_Y(f(y), f(x))}{d_Y(f(y), f(x'))} \leq \eta \left( \frac{d_X(y, x)}{d_X(y, x')} \right).
\]

Bilipschitz embedding if

\[
a \ d(x, y) \leq d(f(x), f(y)) \leq b \ d(x, y).
\]

If we can chose \( a = 1/K \) and \( b = K \) then we say \( f \) is \( K \)-bilipschitz. If we can chose \( a = s/K \) and \( b = sK \) then we say that \( f \) is a \((K, s)\)-quasi-similarity. We say that a group of bilipschitz maps/quasi-similarities
is uniform if \( K \) is uniform over all group elements.

(K,C)-Quasi-isometric embedding if

\[
-C + \frac{1}{K}d_X(x, x') \leq d_Y(f(x), f(x')) \leq Kd_X(x, x') + C
\]

S-Similarity

\[
d_Y(f(x), f(x')) = Sd_X(x, x')
\]

Uniform embedding if for some \( \rho : [0, \infty) \to [0, \infty) \) with \( \rho(t) \to \infty \) as \( t \to \infty \)

\[
\rho(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq Kd_X(x, x') + C
\]

3 \( CAT(-k) \) spaces and their visual boundaries

Let \( X \) be a \( CAT(-k) \) metric space. Riemannian manifolds with sectional curvature \( \leq -k \) and their convex subsets are the prime examples of \( CAT(-k) \) spaces. Gluing two \( CAT(-k) \) spaces along closed convex subsets also results in a \( CAT(-k) \) subspace. Note that up to rescaling the metric by a constant we can assume that \( k = 1 \).

Given a \( CAT(-1) \) space \( X \) and \( a \in \partial X \) and we define the Euclid-Cygan metric on \( \partial X - \{a\} \) as is done in the appendix in [HP97]. Let \( \mathcal{H} \) be a
horosphere centered at $a$ and $b, c \in \partial X -\{a\}$. Let $b_t, c_t$ be two geodesics connecting $b, c$ to $a$ with $b_0, c_0 \in \mathcal{H}$. (Note that this orientation is the opposite of what is given in [HP97]). The Euclid-Cygan metric on $\partial X -\{a\}$ is given by

$$d_{a, \mathcal{H}}(b, c) = \lim_{t \to -\infty} e^{\frac{1}{2}(2t + d_X(b_t, c_t))}.$$  

We can also endow $\partial X -\{a\}$ with a visual parabolic metric. For any $b, c \in \partial X -\{a\}$ the visual parabolic metric is given by

$$\bar{d}_{a, \mathcal{H}}(b, c) = e^{t_0}$$

where $t_0$ is the point at which $d_X(b_{t_0}, c_{t_0}) = 1$. It is easy to see that the two metrics are bilipschitz equivalent since if $t \leq t_0$ then

$$2(t_0 - t) + 1 - C \leq d_X(b_t, c_t) \leq 2(t_0 - t) + 1 + C$$

and so

$$d_{a, \mathcal{H}}(b, c) = \lim_{t \to -\infty} e^{\frac{1}{2}(2t + d_X(b_t, c_t))} \approx \lim_{t \to -\infty} e^{\frac{1}{2}(2t + (t_0 - t) + 1)} \approx e^{t_0} = \bar{d}_{a, \mathcal{H}}(b, c).$$

(See [SX] for more details).

**Remarks.** Note that the reason $e$ is chosen as a base here is because $X$ is $CAT(-1)$. If $X$ were $CAT(-k)$ the base would be $e^{\sqrt{k}}$ instead of $e$. We can in fact choose the base to be $e^\alpha$ for any $\alpha < 1$. This is equivalent to snowflaking the metric.

**Definition 2** For any $b \in \partial X -\{a\}$ there is a unique geodesic $b_t$ connecting $b$ to $a$ with $b_0 \in \mathcal{H}$. We call such geodesic a vertical geodesic.

In this paper we will only be working with $CAT(-k)$ spaces that have the property that for every $x \in X$ there is a vertical geodesic passing through $x$. In this case we can view $X$ as a union (not necessarily disjoint) of all vertical geodesics

$$X = \bigcup_{b \in \partial X -\{a\}} b_t.$$
Definition 3 We endow $X$ with a height function (this is just a horofunction) $h : X \to \mathbb{R}$ given by $h(x) = t$ if $x = b_t$. Note that if $x = b_t$ and $x = b_{t'}$, then necessarily $t = t'$.

A quasi-isometry of $X$ induces a quasisymmetric map

$$\partial f : \partial X - \{a\} \to \partial X - \{f(a)\}$$

with respect to either $d_{a, H}$ or $d_{a, H}$. Since the two metrics are bilipschitz equivalent we will drop the distinction between them and simply write $d_\infty$.

In the following sections we describe all of the CAT$(-1)$ spaces we will be working with.

3.1 Negatively curved homogeneous spaces

Let $N$ be a connected, simply connected, nilpotent Lie group. We say that a one parameter group of automorphisms $\varphi_t$ is contracting (resp. expanding) if for all $g \in N$ we have $\varphi_t(g) \to 1$ (resp. $\varphi_{-t}(g) \to 1$) as $t \to \infty$.

If $\varphi_t : N \to N$ is a one parameter group of expanding automorphisms then $G_\varphi = N \rtimes_\varphi \mathbb{R}$ is a negatively curved homogeneous space when endowed with an appropriate left invariant Riemannian metric [Hei74, CdCMT]. By [Hei74] we have that, up to rescaling the metric, all $N \rtimes_\varphi \mathbb{R}$ can be endowed with a CAT$(-1)$ metric. The geometry of these spaces have been studied by [FM00, Pen11, Ahl02]. Some of their boundaries have been analyzed in [Dym10, DP11, SX, Xiec, Xiea, Xieb].

Since $G_\varphi$ is negatively curved, we can consider its parabolic visual boundary $\partial G_\varphi - \{\infty\}$ where $\infty$ is chosen so that vertical geodesics are be given by

$$\gamma_g(t) = (g, t) \in N \rtimes_\varphi \mathbb{R}.$$ 

We can identify $\partial G_\varphi - \{\infty\}$ with $N$ and then the height function is simply given by $h(g, t) = t$.

To get a better grasp of the metric $G_\varphi$ we define a quasi-isometrically equivalent metric on $G_\varphi$ that is easier to work with than a Riemannian metric. Let $d_N$ be a metric induced by a left invariant metric on $N$ and for each $t \in \mathbb{R}$ set

$$d_t(g_1, g_2) = d_N(\varphi_{-t}(g_1), \varphi_{-t}(g_2)) = \|\varphi_{-t}(g_1^{-1}g_2)\|.$$
Let $\bar{d}$ be the largest metric on $G_{\varphi}$ such that the vertical geodesics given above are actually geodesics and the distance on each level set $h^{-1}(t) = N \times \{ t \} \simeq N$ is $d_t$ [Gro87]. Note also that the level set $N \times \{ t \}$ with the metric $d_t$ is exponentially distorted in $G_\varphi$. With this metric it is easy to verify that for two distinct vertical geodesics $\gamma_{g_0}$ and $\gamma_{g_1}$ we have

$$\lim_{t \to \infty} d(\gamma_{g_0}(-t), \gamma_{g_1}(-t)) = \infty \quad \text{and} \quad \lim_{t \to \infty} d(\gamma_{g_0}(t), \gamma_{g_1}(t)) = 0.$$  

Note that if $d_0(g_0, g_1) = 1$ then $1/K - C \leq d((g_0, t_0), (g_1, t_0)) \leq K + C$ so that up to bilipschitz equivalence we can interpret the parabolic visual metric as

$$d_\infty(g_0, g_1) = e^{t_0}$$

where $t_0$ is the smallest value at which $d_0(g_0, g_1) = 1$. See Section 5 of [SX] for more details.

3.1.1 Snowflaking

It is worthwhile to note that by reparametrizing $\varphi_t$ as $\varphi'_t = \varphi_{\alpha t}$ we get boundary metrics on $G_\varphi$ and $G_{\varphi'}$ that are snowflake equivalent. In particular

$$d_{\infty, \varphi} = d_{\infty, \varphi'}^\alpha.$$  

Note that there is always a range of admissible $\alpha$ that ensure that $d_{\infty, \varphi}^\alpha$ is actually a metric. Alternatively we could have chose the base $a = e^{\alpha}$ instead of $e$ in our definition of boundary metric. This will be important later when we define the millefeuille space (see Section 3.3).

3.2 Trees

Any tree $T$ is a $CAT(-1)$ space so again by fixing a point $\xi$ at infinity we can define the parabolic visual boundary with respect to $\xi$. Picking a point at infinity induces an orientation on edges (towards the point at infinity). This in turn induces a height function: designate a base point vertex to be at height zero then use orientation to determine the heights of all of the other vertices. Vertical geodesics in $T$ are the geodesics that are compatible with the height function. The parabolic visual boundary is again just the set of vertical geodesics and in this case the Euclid-Cygan metric can be interpreted as the $e^{t_0}$ where $t_0$ is the height at which the two vertical geodesics first coincide. Note that for a tree we can define a parabolic visual metric $d^{t_0}$ for any base $a > 1$ and still have it be a metric.
3.3 Millefeuille space

Let $T_{m+1}$ be the regular $m + 1$ valent tree with orientation such that each vertex has $m$ incoming edges and one outgoing edge. With this orientation there is a natural choice for $\infty \in \partial T_{m+1}$. Again, vertical geodesics are given by the coherently oriented infinite geodesics. In this case we can identify $\partial T_{m+1} - \{\infty\}$ with the $m$-adics $\mathbb{Q}_m$ (see [FM98] for the identification).

**Definition 4 (Millefeuille space)** Let $G_\varphi = N \rtimes_\varphi \mathbb{R}$ be a negatively curved homogeneous space with height function $h_\varphi : G_\varphi \to \mathbb{R}$. Let $h_m : T_{m+1} \to \mathbb{R}$ be a height function. The millefeuille space is defined to be

$$X_{N,\varphi,m} = \{(x,y) \in G_\varphi \times T_{m+1} \mid h_\varphi(x) = h_m(y)\}$$

with the induced $L^\infty$ metric.

Alternatively we can view $X_{N,\varphi,m}$ as the metric fibration

$$\pi : X_{N,\varphi,m} \to T_{m+1}$$

where $\pi^{-1}(\ell)$ is identified with $G_\varphi$ for each coherently oriented line $\ell$ in $T_{m+1}$ via a height-preserving isometry.

This space $X_{N,\varphi,m}$ was first defined in Section 7 of [CdCMT]. In that section, it is also noted that $X_{N,\varphi,m}$ is a $\text{CAT}(-k)$ space if $G_\varphi$ is $\text{CAT}(-k)$. This is because locally $X_{N,\varphi,m}$ is obtained by gluing $m$ copies of $G_\varphi$, along closed convex subsets (namely along horoballs of $G_\varphi$).

**Definition 5** Following the terminology coined by Farb-Mosher in [FM00], we call each $\pi^{-1}(\ell)$ a hyperplane of $X$ and each $\pi^{-1}(v)$ a horizontal leaf.

As with $G_\varphi$ and $T_{m+1}$ there is a natural choice of $\infty \in X_{N,\varphi,m}$ such that the vertical geodesics in $X_{N,\varphi,m}$ are precisely the geodesics that project to vertical geodesics in both $G_\varphi$ and $T_{m+1}$. There is also an obvious induced height function $h : X_{N,\varphi,m} \to \mathbb{R}$ given by $h(x,y) = h_\varphi(x) = h_m(y)$.

**Proposition 6** The parabolic visual boundary $\partial X_{N,\varphi,m} - \{\infty\}$ is bilipschitz equivalent to $(N \times \mathbb{Q}_m, d_{\varphi,m})$ where $d_{\varphi,m}$ is the maximum of the metrics $d_\varphi$ (on $N$) and $d_{\mathbb{Q}_m}$ (on $\mathbb{Q}_m$).
Proof. Note that for any two distinct vertical geodesics $\gamma$ and $\gamma'$ we have three possible cases. If $\gamma$ and $\gamma'$ project to the same geodesic in $T_{m+1}$ then $\gamma, \gamma'$ both lie in the same hyperplane $f^{-1}(\ell) \simeq \phi$ in which case we can identify $\gamma, \gamma'$ with vertical geodesics in $G$ (namely $\gamma \simeq \gamma_{g_0}$ and $\gamma' \simeq \gamma_{g_1}$ for some $g_0, g_1 \in N$). Then

$$d_{\infty,X}(\gamma, \gamma') = d_{\infty,G}(g_0, g_1).$$

Likewise, if the projection of $\gamma$ and $\gamma'$ to $G$ is the same then the two geodesics coincide above some $t_0 \in \mathbb{Z}$. In this case $\gamma$ and $\gamma'$ lie in the same copy of $T_{m+1}$ and so we have

$$d_{\infty,X}(\gamma, \gamma') = d_{\infty,T_{m+1}}(\gamma, \gamma').$$

Finally the last case is when $\gamma$ and $\gamma'$ project to two different vertical geodesics in both factors. Nevertheless, eventually (above height $t_1$), these two geodesics lie in the same hyperplane $f^{-1}(\ell) \simeq \phi$. Then, above $t_1$, we can identify $\gamma \simeq \gamma_{g_0}$ and $\gamma' \simeq \gamma_{g_1}$ for some $g_0, g_1 \in N$. If $d_{t_1}(\gamma(t_1), \gamma'(t_1)) \geq 1$ then

$$d_{\infty,X}(\gamma, \gamma') = d_{\infty,G}(g_0, g_1)$$

since then the height at which $\gamma, \gamma'$ are distance one is above $t_1$. Otherwise, $d_{t_1}(\gamma(t_1), \gamma'(t_1)) < 1$ and the boundary metric has the property that

$$1/K d_{\infty,T_{m+1}}(\gamma, \gamma') \leq d_{\infty,X}(\gamma, \gamma') \leq K d_{\infty,T_{m+1}}(\gamma, \gamma').$$

Finally, in order to get the standard metric on $Q_m$ (i.e. $d_{\infty,T_{m+1}}(\gamma, \gamma') = m^{t_0}$ where $t_0$ is the height at which $\gamma, \gamma'$ initially come together) we must snowflake our boundary metric by $\alpha = \ln m$. To ensure that is possible we might have to replace $\phi(t)$ with $\phi'(t) = \phi(at)$. (See the comments at the end of Section 3.1).

3.4 Quasi-isometries

Definition 6. Let $X$ be a CAT($-1$) space with height function $h : X \to \mathbb{R}$. We say that a quasi-isometry $f : X \to X$ is height-respecting if there is a constant $A$ such that $f$ maps any height level set of $h$ to within distance $A$ of a height level set and if the map induced on height is bounded distance from a translation. In other words there exists a constant $a$ such that if $h(x) = t$ then

$$-C + t + a \leq h(f(x)) \leq t + a + C.$$
It is now a well known fact (Lemma 6.1 [Xiea] and [FM98]) that for $G_\varphi$ and $T_{m+1}$ height-respecting quasi-isometries (up to bounded distance) are in one-to-one correspondence with bilipschitz maps of the parabolic visual boundary. Combining these two facts we can see that the same is true for $X_{N,\varphi,m}$.

**Proposition 7** Any height-respecting quasi-isometry of $X_{N,\varphi,m}$ induces a bilipschitz maps of the parabolic visual boundary

$$\partial X_{N,\varphi,m} - \{\infty\} \simeq N \times \mathbb{Q}_m.$$ 

**Remark.** Note that while we prove here that all quasi-isometries of $X_{N,\varphi,m}$ are height-respecting it is not always the case that all quasi-isometries of $T_{m+1}$ and $G_\varphi$ are height-respecting. This is clear for $T_{m+1}$ however for $G_\varphi$ the answer is more subtle. For instance when $\varphi_t(x) = e^tx$ the space $\mathbb{R}^n \rtimes \varphi \mathbb{R}$ is isometric to $\mathbb{H}^{n+1}$ whose quasi-isometries can be identified with the quasiconformal maps of $S^n$. The other rank one symmetric spaces can also be written as $N \rtimes \varphi \mathbb{R}$ for the appropriate $N$ and $\varphi$. In [SX, Xiec, Xiea], Xie showed that when $\mathbb{R}^n \rtimes \varphi \mathbb{R}$ is not isometric to $\mathbb{H}^{n+1}$ then all quasi-isometries are height-respecting. In [Xieb], he was able to show that the same result is true for certain $N \rtimes \varphi \mathbb{R}$. It is an open question whether all quasi-isometries of negatively curved homogeneous spaces that are not isometric to symmetric spaces are height-respecting.

### 4 Proof of Theorem 1

In this section we show that any quasi-isometry of $X_{N,\varphi,m}$ is height-respecting. We start with Farb-Mosher’s Theorem 7.7 [FM00].

**Theorem 8 (Theorem 7.7 in [FM00])** Let $\pi : X \to T$ be a metric fibration over a bushy tree $T$ such that the fibers of $\pi$ are contractible $k$-manifolds for some $k$. Let $f : X \to X$ be a self quasi-isometry. Then there exists a constant $A$, depending only on the metric fibration data of $\pi$, the quasi-isometry data of $f$ and $T$ such that

1. For each hyperplane $P \subset X$ there exists a unique hyperplane $Q \subset X$ such that $d_H(f(P), Q) \leq A$.  

10
2. For each horizontal leaf $L \subset X$ there is a horizontal leaf $L'$ such that $d_H(f(L), L') \leq A$.

We apply this theorem to $X = X_{N,\varphi,m}$ and $T = T_{m+1}$. Note that there are two possible types of hyperplanes in $X_{N,\varphi,m}$. If $\ell$ is a coherently oriented geodesic in $T_{m+1}$ then $f^{-1}(\ell) \simeq G_\varphi$. Otherwise, $\ell$ changes orientation exactly once (say at $t_0$) so $f^{-1}(\ell) \simeq H$ is a doubled horoball complement (i.e. the union of two horoball complements $\{(g,t) \mid t \leq t_0\} \subset G_\varphi$ identified along the horosphere $N \times \{t_0\}$.) Note also that if we endow $H$ with the induced metric from $X$ then $H$ is not a geodesic metric space. For instance, the points $p = (g_0, t_0)$ and $q = (g_1, t_0)$ are not connected by a geodesic. In fact, the induced path metric distorts the distances between $p$ and $q$ exponentially. We will use this fact to prove the following proposition.

**Proposition 9** A quasi-isometry $f: X_{N,\varphi,m} \to X_{N,\varphi,m}$ takes coherently oriented hyperplanes to coherently oriented hyperplanes.

**Proof.** Let $P \subset X_{N,\varphi,m}$ be a coherently oriented hyperplane. Suppose

$$f(P) \simeq Q$$

is a doubled horoball complement. Note that $f$ maps horizontal leaves in $P$ to within distance $A$ of horizontal leaves in $Q$. Let $p', q'$ be two points that are mapped to within distance $C$ of $p = (g_0, t_0)$ and $q = (g_1, t_0)$. Without loss of generality, we can assume that $p' = (g_0', t_0')$ and $q' = (g_1', t_0')$ for some $t_0'$. Recall that by slightly enlarging the quasi-isometry constants we can assume that $f$ and its coarse inverse $\tilde{f}$ are actually Lipschitz (see for example [FM98]). This implies that $f$ (and $\tilde{f}$) sends rectifiable paths of length $L$ to rectifiable paths of length at most $KL$.

Note that any path connecting $p$ and $q$ that lies in a $C$-neighborhood of $Q$ in $X$ has length at least $K' \exp(d(p,q))$ for some $K' > 0$. Note also that if $d_P$ denotes the distance in $P$ then the geodesic $\gamma$ between $p'$ and $q'$ has length

$$d(p', q') = d_P(p', q')$$

since $P \subset X_{N,\varphi,m}$ is geodesically embedded. Now $\text{len}(\gamma) = d(p', q')$ and $d(p', q') \leq Kd(p, q) + C''$ so

$$\text{len}(f(\gamma)) \leq K\text{len}(\gamma) \leq K^2 d(p, q) + C''.$$
But $f(\gamma)$ must lie in a $C$-neighborhood of $Q$ so
\[
\text{len}(f(\gamma)) \geq K' \exp(d(p,q)).
\]
This implies that
\[
K' \exp(d(p,q)) \leq K^2 d(p,q) + C''
\]
which is not possible if $d(p,q)$ is large enough.  

Now we know that for each coherently oriented $P \subset X_{N,\varphi,m}$, the quasi-isometry $f$ induces a quasi-isometry of $G_{\varphi}$ that permutes height level sets. From Proposition 5.8 in [FM00] and Theorem 33 in [Ahl02] we have that any quasi-isometry of $G_{\varphi}$ that permutes height level sets is height-respecting where the constants $A, a, C'$ in Definition 6 depend only on $G_{\varphi}$ and the quasi-isometry constants. This in turn implies that $f$ induces a height-respecting map of $T_{m+1}$ and more generally of $X_{N,\varphi,m}$ itself. Finally, we appeal to Proposition 7 to see that this implies that any quasisymmetric map of $Y \simeq \partial X$ is actually bilipschitz.

\textbf{Remark.} Theorem 8 (Theorem 7.7 from [FM00]) relies on Theorem 7.3 from [FM00] which we state below. As noted in the introduction, Theorem 1 should hold for the boundaries of any anti-horocyclic product of a tree with a negatively curved space satisfying Theorems 7.3 and 7.7.

\textbf{Theorem 10 (Theorem 7.3 in [FM00])} Let $\pi : X \to T$ be a metric fibration whose fibers are contractible $k$-manifolds for some $k$. Let $P$ be a contractible $(k + 1)$-manifold which is a uniformly contractible, bounded-geometry, metric simplicial complex. Then for any uniformly proper embedding $\phi : P \to X$, there exists a unique hyperplane $Q \subset X$ such that $\phi(P)$ and $Q$ have finite Hausdorff distance in $X$. The bound on Hausdorff distance depends only on the metric fibration data for $\pi$, the uniform contractibility data and bound geometry data for $P$, and the uniform properness data for $\phi$. 

12
\section{Uniform subgroups of $\text{Bilip}_M(\mathbb{R}^n \times \mathbb{Q}_m)$}

Let $N \simeq \mathbb{R}^n$ and $\varphi = \varphi_M$ where $\varphi_M(v) = Mv$ for some $M \in GL_n(\mathbb{R})$ with all eigenvalues of norm greater than one. Without loss of generality we can assume that $M$ is in absolute Jordan form. Suppose $e^{\alpha_i}$ for $1 \leq i \leq r$ are the eigenvalues of $M$. In case $M$ is diagonal, we give an explicit form for the visual metric on $\mathbb{R}^n \simeq \partial G_\varphi$. Let $v, w \in \mathbb{R}^n$ and $|\Delta x_i|$ the norm of the difference of $v$ and $w$ in the $e^{\alpha_i}$ eigenspace. Then a parabolic visual metric can be given by

$$D_M(v, w) := d_\infty(v, w) = \max_i \{|\Delta x_i|^{\alpha_i}\}.$$ 

In case $M$ is not diagonalizable the metric $D_M$ is more cumbersome to write down so we refer the reader to [DP11] and [Xiea].

Note that by Proposition 6, a parabolic visual metric on the boundary of $X_{\mathbb{R}^n, \varphi, m}$ can be given by

$$D_{M,m}((v, y), (v', y')) = \max\{D_M(v, v'), d_{\mathbb{Q}_m}(y, y')\}$$

where $d_{\mathbb{Q}_m}$ is the standard metric on the $m$-adics. In this section we study subgroups of bilipschitz maps of $\mathbb{R}^n \times \mathbb{Q}_m$ with respect to the metric $D_{M,m}$.

\textbf{Notation 1} Denote by

- $\text{Bilip}_M(\mathbb{R}^n \times \mathbb{Q}_m)$ the set of all bilipschitz maps of $\mathbb{R}^n \times \mathbb{Q}_m$ with respect to the metric $D_{M,m}$,
- $\text{Sim}_M(\mathbb{R}^n \times \mathbb{Q}_m)$, the set of similarities with respect to $D_{M,m}$,
- $\text{ASim}_M(\mathbb{R}^n \times \mathbb{Q}_m)$ the set of similarities composed with almost translations (i.e. maps in $\text{Bilip}_M(\mathbb{R}^n \times \mathbb{Q}_m)$ of the form

$$ (x_1, x_2, \cdots, x_r, y) \mapsto (x_1 + B_1(x_2, \cdots, x_r, y), \cdots, x_r + B_r(y), y).$$

Here the $B_i$ are $\frac{\alpha_i}{\alpha_j}$-Hölder in each $x_j$ and Lipschitz in $y$).

For our purposes we will call a subgroup of bilipschitz maps uniform if it is uniform as a group of quasi-similarities (see Section 2).
Theorem 11 Let $U$ be a uniform separable subgroup of $\text{Bilip}_{M}(\mathbb{R}^n \times \mathbb{Q}_m)$ that acts cocompactly on the space of distinct pairs of points of $\mathbb{R}^n \times \mathbb{Q}_m$. Then $U$ can be conjugated by a bilipschitz map into $\text{ASim}_M(\mathbb{R}^n \times \mathbb{Q}_p)$ for some $p$.

Before beginning the proof of this theorem it is worth mentioning that after some initial setup this proof follows very closely the proofs of Theorem 2 in [Dym10] and Theorem 2 in [DP11]. For this reason we will provide only an outline of the proof and fill in the details of where the proofs differ. Furthermore when we do need to refer to the specifics of $D_M$, for simplicity we will only refer to the diagonal case. What is new in this Theorem is the introduction of the $\mathbb{Q}_n$ coordinate. Note also that $p$ might be different from $m$.

Lemma 12 If $f \in \text{Bilip}_{M}(\mathbb{R}^n \times \mathbb{Q}_m)$ then $f$ decomposes as

$$f(x,y) = (f_1(x,y), f_2(y))$$

where $f_1$ is bilipschitz with respect to $D_M$ in the $x$ coordinate and Lipschitz in $y$ with respect to $d_{\mathbb{Q}_M}$. The map $f_2$ is bilipschitz in $y$ with respect to $d_{\mathbb{Q}_M}$.

Proof. In this case the proof is simply topological. Since the connected components of $\mathbb{R}^n \times \mathbb{Q}_m$ are given by $\mathbb{R}^n \times \{y_0\}$ for each $y_0 \in \mathbb{Q}_m$ the map $f$ maps $\mathbb{R}^n \times \{y_0\}$ to $\mathbb{R}^n \times \{y_1\}$ and induces a bilipschitz map between these two sets. $
$

Note that combining this with the structure of $\text{Bilip}_{D_M}$ maps from Proposition 4 in [Dym10] we see that $f$ has the form

$$f(x_1, \ldots, x_r, y) = (f_1(x_1, \ldots, x_r, y), \ldots, f_r(x_r, y), g(y))$$

where $f_i$ are $\frac{\alpha}{\alpha_j}$-Hölder in each $x_j$ and Lipschitz in $y$.

Theorem 13 (Theorem 7 in [MSW03]) Suppose that $U \subseteq \text{Bilip}(\mathbb{Q}_m)$ is a uniform subgroup. Suppose in addition that the induced action of $U$ on the space of distinct pairs in $\mathbb{Q}_m$ is cocompact. Then there exists $p \geq 2$ and a bilipschitz homeomorphism $\mathbb{Q}_m \mapsto \mathbb{Q}_p$ which conjugates $H$ into the similarity group $\text{Sim}(\mathbb{Q}_p)$. 

14
Proof. (Of Theorem 5) By Lemma 12 we have a natural projection
\[ \pi : \text{Bilip}_M(\mathbb{R}^n \times \mathbb{Q}_m) \to \text{Bilip}(\mathbb{Q}_m). \]

A uniform subgroup \( U \subseteq \text{Bilip}_M(\mathbb{R}^n \times \mathbb{Q}_m) \) projects to a uniform subgroup \( \pi(U) \) of \( \text{Bilip}(\mathbb{Q}_m) \). Since \( U \) was assumed to act cocompactly on pairs of points in \( \mathbb{R}^n \times \mathbb{Q}_m \) we have that \( \pi(U) \) acts cocompactly on pairs of \( \mathbb{Q}_m \). By Theorem 7 in [MSW03] (Theorem 13 above) we can conjugate \( \pi(U) \) via a bilipschitz map into \( \text{Sim}(\mathbb{Q}_p) \) for some \( p \) possibly different from \( m \).

After this conjugation we can assume that \( U \) is a uniform subgroup of \( \text{Bilip}_M(\mathbb{R}^n \times \mathbb{Q}_p) \) where \( f \in U \) acts by
\[ f(x, y) = (f_1(x_1, \ldots, x_r, y), f_2(x_2, \ldots, x_r, y), \ldots, f_r(x_r, y), \sigma_f(y)) \]
and \( \sigma_f \) is a similarity of \( \mathbb{Q}_p \) and for each fixed \( y \) the map
\[ f_y(x) = (f_1(x_1, \ldots, x_r, y), f_2(x_2, \ldots, x_r, y), \ldots, f_r(x_r, y)) \]
is bilipschitz with respect to \( D_M \). We have further projections
\[ \pi_i : \text{Bilip}_M(\mathbb{R}^n \times \mathbb{Q}_p) \to \text{Bilip}_{M_i}(\mathbb{R}^{n_i} \times \mathbb{Q}_p) \]
given by \( \pi_i(f) = (f_i, \ldots, f_r, \sigma_f) \). For each fixed \( y \in \mathbb{Q}_p \) we have \( \pi_i(f)_y = (f_i, \ldots, f_r) \) is a bilipschitz map with respect to \( D_{M_i} \) where \( M_i \) is the submatrix of \( M \) that contains all eigenvalues greater than or equal to \( \alpha_i \).

Now we can follow the conjugation proof found in Section 3 of [Dym10]. The proof is easily adapted to our situation. The key change is that whenever a measure is called for we use the product \( \mu \) of Lebesgue measure \( \lambda \) on \( \mathbb{R}^n \) and Hausdorff measure \( \nu \) on \( \mathbb{Q}_p \). Following this proof, in the appendix, we will reprove any Lemmas that rely on measure theoretic arguments.

The conjugation is done inductively on \( i \) starting at \( i = r \) and working down to \( i = 1 \). So for induction we assume that we have a uniform subgroup \( U \subseteq \text{Bilip}_{M_i}(\mathbb{R}^{n_i} \times \mathbb{Q}_p) \) where each \( f \in U \) has the form
\[ f(x_i, \ldots, x_r, y) = (f_i(x_i, \ldots, x_r, y), F(x_{i+1}, \ldots, x_r, y), \sigma_f(y)) \]
where \( (F, \sigma_f) \in \text{ASim}_{M_{i+1}}(\mathbb{R}^{n_{i+1}} \times \mathbb{Q}_p) \). If \( x_i \) represents a one dimensional subspace then Section 3.3 in [Dym10] goes through unchanged. If \( x_i \) represents a higher dimensional subspace then we follow Section 3.4. First we find a \( U \) invariant foliated conformal structure i.e. a measurable assignment
\[ \eta : \mathbb{R}^n \times \mathbb{Q}_p \to \text{SL}(s_i, \mathbb{R})/\text{SO}(s_i, \mathbb{R}) \]
such that
\[ \eta(x_i, \ldots, x_r, y) = f'_i(x_i, \ldots, x_r, y)[\eta(f(x_i, \ldots, x_r, y))] \]
where the derivative of \( f_i \) is taken in the \( x_i \) coordinate and the action is a standard matrix action on \( SL(s_i, \mathbb{R})/SO(s_i, \mathbb{R}) \) (see Section 3.4 in [Dym10] for more details). Again this only uses differentiability in the \( x_i \) direction and the group structure so no changes are needed in the proof.

Next we follow Theorem 13 in [Dym10] and prove that if \( \eta \) is approximately continuous at some radial point then we can find a conjugating map that conjugates \( f_i \) to be a similarity in the \( x_i \) coordinate while preserving the fact that \( F \in ASim_{M_i+1}(\mathbb{R}^{n_i} \times \mathbb{Q}_p) \). Note that by Theorem 2.9.13 in [Fed69] we have that any measurable map from \( \mathbb{R}^n \times \mathbb{Q}_p \) to a separable metric space is approximately continuous almost everywhere. In particular \( \eta \) is approximately continuous almost everywhere. Furthermore since \( U \) acts cocompactly on pairs of points of \( \mathbb{R}^n \times \mathbb{Q}_p \) every point of \( \mathbb{R}^n \times \mathbb{Q}_p \) is a radial point.

The conjugating map is constructed from a sequence of \( g_i \in U \) (the same sequence used in defining the radial point) rescaled by a family of similarities in \( Sim_{M_i}(\mathbb{R}^{n_i} \times \mathbb{Q}_p) \) following [Dym10]. The proof that the conjugation does what we claim it does (i.e. that \( f_i \) is a similarity in \( x_i \)) requires a couple of lemmas on quasi-conformal maps. These are found in Section 3.4.2 in [Dym10]. We restate and comment on one of these following this proof (Lemma 14 below).

After the conjugation, each \( f \in U \) has the form
\[ f(x_i, \ldots, x_r, y) = (c_{x_{i+1}, \ldots, x_r, y}A_{x_{i+1}, \ldots, x_r, y}(x_{i+1}, \ldots, x_r, y), F(x_{i+1}, \ldots, x_r, y), \sigma_f(y)) \]
and we must show that \( c \in \mathbb{R} \) and \( A \in O(s_i, \mathbb{R}) \) do not depend on \( x_{i+1}, \ldots, x_r, y \). This follows sections 3.5 and 3.6 in [Dym10].

**Lemma 14 (Lemma 10 in [Dym10])** Let \( F_{K,N} \) be a family of \((N,K)\)-bilipschitz (quasi-similarity) maps in \( Bilip_{M_i}(\mathbb{R}^n \times \mathbb{Q}_p) \) such that
\[ f(x_i, \ldots, x_r, y) = (f_i(x_i, \ldots, x_r, y), F(x_{i+1}, \ldots, y), \sigma_f(y)) \]
where \( f_i \) is bilipschitz in \( x_i \) and \( (F, \sigma_f) \) is in \( ASim_{M_i+1}(\mathbb{R}^{n_i+1} \times \mathbb{Q}_p) \). Then there exist \( b, b' \) such that for any measurable set \( E \subseteq \mathbb{R}^{n_i} \times \mathbb{Q}_p \) and all \( f \in F_K \)
\[ b\mu(E) \leq \mu(f(E)) \leq b'\mu(E). \]
Proof. Note that if $\sigma_f$ is an $N$-similarity then $\sigma_f$ distorts measure by a factor of $N$. Otherwise this lemma is proved using Fubini’s Theorem in the same way as the corresponding Lemma 10 in [Dym10].

References


[Xiea] Xiangdong Xie, *Large scale geometry of negatively curved $\mathbb{R}^n \times \mathbb{R}$*, preprint.
